

# On $L_p$ - theory for stochastic parabolic integro-differential equations

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## Abstract

The existence and uniqueness in fractional Sobolev spaces of the Cauchy problem to a stochastic parabolic integro-differential equation is investigated. A model problem with coefficients independent of space variable is considered. The equation arises in a filtering problem with a jump signal and jump observation process.

## 1 Introduction

Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a complete probability space with a filtration of  $\sigma$ -algebras  $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$  satisfying the usual conditions. Let  $\mathcal{R}(\mathbb{F})$  be the progressive  $\sigma$ -algebra on  $[0, \infty) \times \Omega$ . Let  $(U, \mathcal{U}, \Pi)$  be a measurable space with a  $\sigma$ -finite measure  $\Pi$ ,  $E = [0, T] \times \mathbf{R}_0^d$ ,  $\mathbf{R}_0^d = \mathbf{R}^d \setminus \{0\}$ .

Let  $p^{(\alpha)}(dt, dy)$ ,  $\alpha \in (0, 2)$ , and  $\nu(dt, dv)$  be  $\mathbb{F}$ -adapted point measures on  $([0, \infty) \times \mathbf{R}_0^d, \mathcal{B}([0, \infty)) \otimes \mathcal{B}(\mathbf{R}_0^d))$  and  $([0, \infty) \times U, \mathcal{B}([0, \infty)) \otimes \mathcal{U})$  with compensators  $l^{(\alpha)}(t, y)dydt/|y|^{d+\alpha}$  and  $\Pi(dv)dt$ . We assume that the measures  $\nu$  and  $p^{(\alpha)}$ ,  $\alpha \in (0, 2)$ , have no common jumps.

For fixed  $\alpha \in (0, 2]$ , we consider the linear stochastic integro-differential parabolic equation

$$\begin{aligned} du(t, x) = & (A^{(\alpha)}u(t, x) - \lambda u(t, x) + f(t, x))dt \\ & + \int_{\mathbf{R}_0^d} [u(t-, x + y) - u(t-, x) + g(t, x, y)]q^{(\alpha)}(dt, dy)1_{\alpha \in (0, 2)} \\ & + [1_{\alpha=2}\sigma^i(t)\partial_i u(t, x) + h(t, x)]dW_t + \int_U \Phi(t, x, v)\eta(dt, dv) \quad \text{in } E, \\ u(0, x) = & u_0(x) \text{ in } \mathbf{R}^d, \end{aligned} \tag{1}$$

where  $\lambda \geq 0$ ,  $W_t$  is a cylindrical  $\mathbb{F}$ -adapted Wiener process in a separable Hilbert space  $Y$  and  $q^{(\alpha)}$ ,  $\alpha \in (0, 2)$ ,  $\eta$  are martingale measures defined by

$$q^{(\alpha)}(dt, dy) = p^{(\alpha)}(dt, dy) - l^{(\alpha)}(t, y) \frac{dy dt}{|y|^{d+\alpha}}$$

and

$$\eta(dt, dv) = \nu(dt, dv) - \Pi(dv)dt.$$

The input functions  $u_0, f, g, \Phi, h$  satisfy the following measurability assumptions:  $u_0$  is  $\mathcal{F}_0 \otimes \mathcal{B}(\mathbf{R}^d)$ -measurable,  $f$  is  $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbf{R}^d)$ -measurable,  $g$  is  $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbf{R}^d)$ -measurable,  $\Phi$  is  $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{U}$ -measurable and  $h$  is  $Y$ -valued and  $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbf{R}^d)$ -measurable. The operator  $A^{(\alpha)}$  is defined as

$$\begin{aligned} A^{(\alpha)}u(t, x) &= \int_{\mathbf{R}_0^d} \nabla_y^\alpha u(t, x) m^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}} 1_{\alpha \in (0, 2)} \\ &\quad + (b(t), \nabla u(t, x)) 1_{\alpha=1} + \frac{1}{2} B^{ij}(t) \partial_{ij}^2 u(t, x) 1_{\alpha=2}, \end{aligned} \quad (2)$$

where

$$\nabla_y^\alpha u(t, x) = u(t, x + y) - u(t, x) - (\nabla u(t, x), y) \chi^{(\alpha)}(y)$$

with  $\chi^{(\alpha)}(y) = 1_{\alpha \in (1, 2)} + 1_{|y| \leq 1} 1_{\alpha=1}$ . Here and throughout the paper we use the standard convention of summation over repeating indices. The integral part

$$\int_{\mathbf{R}_0^d} \nabla_y^\alpha u(t, x) m^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}} = \Delta^{\alpha/2} u$$

is the fractional Laplacian if  $m^{(\alpha)} = 1$ .

The coefficients  $m^{(\alpha)}(t, y)$  and  $l^{(\alpha)}(t, y)$  are  $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbf{R}_0^d)$ -measurable bounded non-negative functions,  $\sigma^i(t)$ ,  $i = 1, \dots, d$ , are  $\mathcal{R}(\mathbb{F})$ -measurable bounded  $Y$ -valued functions,  $b(t) = (b^1(t), \dots, b^d(t))$  is a  $\mathcal{R}(\mathbb{F})$ -measurable bounded function and  $B(t) = (B^{ij}(t), i, j = 1, \dots, d)$  is a  $\mathcal{R}(\mathbb{F})$ -measurable bounded symmetric non-negative definite matrix-valued function. We assume **parabolicity** of (1), i.e.  $m^{(\alpha)} - l^{(\alpha)} \geq 0$  if  $\alpha \in (0, 2)$  and the matrix  $B^{ij}(t) - \frac{1}{2} \sigma^i(t) \cdot \sigma^j(t)$  is non-negative definite if  $\alpha = 2$  ( $\cdot$  denotes the inner product in  $Y$ ).

The equation (1) is the model problem for the Zakai equation (see [20]) arising in the nonlinear filtering problem. Let  $\alpha \in (0, 2)$  and  $Z_t^i, t \geq 0, i = 1, 2$ , be two independent  $\alpha$ -stable processes defined by

$$Z_t^i = \int_0^t \int_{\mathbf{R}_0^d} y \chi^{(\alpha)}(y) q^{Z^i}(ds, dy) + \int_0^t \int_{\mathbf{R}_0^d} y [1 - \chi^{(\alpha)}(y)] p^{Z^i}(ds, dy),$$

where  $p^{Z^i}(ds, dy)$  is the jump measure of  $Z^i$  and

$$q^{Z^i}(ds, dy) = p^{Z^i}(ds, dy) - m_i^{(\alpha)}(s, y) \frac{dy ds}{|y|^{d+\alpha}}$$

is the martingale measure. Assume that the signal process

$$X_t = X_0 + Z_t^1 + Z_t^2, \quad t \geq 0,$$

and we observe  $Y_t = Z_t^2$ . Suppose that  $X_0$  has a probability density function  $u_0(x)$  and does not depend on  $Z^i, i = 1, 2$ . Then for every function  $f \in C_0^\infty(\mathbf{R}^d)$ , the optimal mean square estimate for  $f(X_t)$ ,  $t \in [0, T]$ , given the past of the observations  $\mathcal{F}_t^Y = \sigma(Y_s, s \leq t)$ , is of the form  $\pi_t(f) = \mathbf{E}(f(X_t) | \mathcal{F}_t^Y)$ . According to [6],

$$\begin{aligned} d\pi_t(f) &= \int \pi_t(f(\cdot + y) - f) q^Y(dt, dy) \\ &+ \int_{\mathbf{R}_0^d} \pi_t(\nabla_y^\alpha f(\cdot)) [m_1^{(\alpha)}(t, y) + m_2^{(\alpha)}(t, y)] \frac{dy}{|y|^{d+\alpha}} dt. \end{aligned}$$

Assume there is a smooth  $(\mathcal{F}_{t+}^Y)$ -adapted filtering density function  $v(t, x)$ ,

$$\mathbf{E}[f(X_t) | \mathcal{F}_t^Y] = \int v(t, x) f(x) dx, \quad f \in C_0^\infty(\mathbf{R}^d).$$

Integrating by parts, we get

$$\begin{aligned} dv(t, x) &= \int [v(t, x + y) - v(t, x)] q^{-Y}(dt, dy) \\ &+ \int_{\mathbf{R}_0^d} \nabla_y^\alpha v(t, x) [m_1^{(\alpha)}(t, -y) + m_2^{(\alpha)}(t, -y)] \frac{dy}{|y|^{d+\alpha}}, \\ v(0, x) &= u_0(x). \end{aligned}$$

The general Cauchy problem for a linear parabolic SPDE of the second order

$$\begin{cases} du = (\frac{1}{2} a^{ij} \partial_{ij} u + b^i \partial_i u + c u + f) dt + (\sigma^i \partial_i u + h u + g) dW_t & \text{in } E, \\ u(0, x) = 0 & \text{in } \mathbf{R}^d \end{cases} \quad (3)$$

driven by a Wiener process  $W_t$  has been studied by many authors. When the matrix  $(a^{ij} - \sigma^i \cdot \sigma^j)$  is uniformly non-degenerate there exists a complete theory in Sobolev spaces and in the spaces of Bessel potentials  $H_s^p(\mathbf{R}^d)$  (see [9] and references therein).

In [3], the equation (1) was considered in fractional Sobolev and Besov spaces in the case of  $A^{(\alpha)} = \Delta^{\alpha/2}$  with  $q^{(a)} = 0, \eta = 0, \sigma = 0$  and a finite dimensional  $Y$ .

In [8], the equation (1) was considered in fractional Sobolev spaces in the following special form (see equation (3.4) in [8]):

$$du = (a(t)\Delta^{\alpha/2} + f)dt + \sum_{k=1}^{\infty} h^k dW_t^k + \sum_{k=1}^{\infty} g^k dY^k, \quad (4)$$

where  $a(t) \geq \delta > 0$  is a positive scalar function,  $W^k$  are independent standard Wiener processes,

$$Y_t^k = \int_0^t \int z [N^k(ds, dz) - \pi_k(dz)ds], t \geq 0, k \geq 1,$$

are independent  $\mathbf{R}^m$ -valued with independent Poisson point measures  $N^k(ds, dz)$  on  $[0, \infty) \times \mathbf{R}_0^m$ ,  $\mathbf{E}N^k(ds, dz) = \pi_k(dz)ds$  and

$$\int |z|^2 \pi_k(dz) < \infty, k \geq 1.$$

Since  $Y_t^k$  are independent they do not have common jump moments and we can introduce a point measure  $\nu(ds, dv)$  on  $[0, \infty) \times U$  with  $U = \mathbf{N} \times \mathbf{R}_0^m$  ( $\mathbf{N} = \{1, 2, \dots\}$ ) by

$$\nu(ds, dv) = \nu(ds, dkdz) = N^k(ds, dz)dk,$$

where  $dk$  is the counting measure on  $\mathbf{N}$ . Then  $\mathbf{E}p(ds, dkdz) = \pi_k(dz)dkds$  and

$$\eta(ds, dv) = \nu(ds, dkdz) - \pi_k(dz)dkds$$

is a martingale measure. Therefore with  $Y = l^2$  (the space of square summable sequences) we can rewrite (4) as

$$du = (a(t)\Delta^{\alpha/2}u + f)dt + h dW_t + \int_U \Phi(s, v)\eta(ds, dv) \quad (5)$$

where  $\Phi(s, v) = g(s, k, z) = g^k(s) \cdot z$ . Thus (4) is a partial case of (1) with  $q^{(\alpha)} = 0$  and  $m^{(\alpha)}(t, y) = a(t)$ . Theorem 5 below shows that the estimates of the main Theorem 3.6 in [8] are not sharp and the assumptions can be relaxed. Contrary to the case of a partial differential equation, in order to handle an equation with  $A^{(\alpha)}$ , it is not sufficient to consider an equation

with fractional Laplacian like (5). Since only measurability of  $m^{(\alpha)}(t, y)$  in  $y$  is assumed, in general (for  $\alpha \in (0, 2)$ ) the symbol of  $A^{(\alpha)}$

$$\psi^{(\alpha)}(t, \xi) = \int \left[ e^{i(\xi, y)} - 1 - \chi_\alpha(y) i(\xi, y) \right] m^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}} - i(b(t), \xi) 1_{\alpha=1}$$

is not smooth in  $\xi$ . In addition,  $m^{(\alpha)}(t, y)$  can degenerate on a substantial set (see Assumption A and Remark 1 below). The equation (1) in Hölder classes was considered in [12].

In this paper, we prove the solvability of the general Cauchy model problem (1) in fractional Sobolev spaces. In Section 2, we introduce the notation and state our main results. In Section 3, we prove some auxiliary results concerning approximation of the input functions. In Section 4, we consider a partial case of (1) with  $q^{(\alpha)} = 0$ , non-random  $m^{(\alpha)}$  and smooth input functions. In the last two sections we give the proofs of the main results.

## 2 Notation, function spaces and main results

### 2.1 Notation

The following notation will be used in the paper.

Let  $\mathbf{N}_0 = \{0, 1, 2, \dots\}$ ,  $\mathbf{R}_0^d = \mathbf{R}^d \setminus \{0\}$ . If  $x, y \in \mathbf{R}^d$ , we write

$$(x, y) = \sum_{i=1}^d x_i y_i, \quad |x| = \sqrt{(x, x)}.$$

We denote by  $C_0^\infty(\mathbf{R}^d)$  the set of all infinitely differentiable functions on  $\mathbf{R}^d$  with compact support.

We denote the partial derivatives in  $x$  of a function  $u(t, x)$  on  $\mathbf{R}^{d+1}$  by  $\partial_i u = \partial u / \partial x_i$ ,  $\partial_{ij}^2 u = \partial^2 u / \partial x_i \partial x_j$ , etc.;  $Du = \nabla u = (\partial_1 u, \dots, \partial_d u)$  denotes the gradient of  $u$  with respect to  $x$ ; for a multiindex  $\gamma \in \mathbf{N}_0^d$  we denote

$$D_x^\gamma u(t, x) = \frac{\partial^{|\gamma|} u(t, x)}{\partial x_1^{\gamma_1} \dots \partial x_d^{\gamma_d}}.$$

For  $\alpha \in (0, 2]$  and a function  $u(t, x)$  on  $\mathbf{R}^{d+1}$ , we write

$$\partial^\alpha u(t, x) = -\mathcal{F}^{-1} [|\xi|^\alpha \mathcal{F} u(t, \xi)](x),$$

where

$$\mathcal{F} h(t, \xi) = \int_{\mathbf{R}^d} e^{-i(\xi, x)} h(t, x) dx, \quad \mathcal{F}^{-1} h(t, \xi) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{i(\xi, x)} h(t, \xi) d\xi.$$

The letters  $C = C(\cdot, \dots, \cdot)$  and  $c = c(\cdot, \dots, \cdot)$  denote constants depending only on quantities appearing in parentheses. In a given context the same letter will (generally) be used to denote different constants depending on the same set of arguments.

## 2.2 Function spaces

Let  $\mathcal{S}(\mathbf{R}^d)$  be the Schwartz space of smooth real-valued rapidly decreasing functions. Let  $V$  be a Banach space with a norm  $|\cdot|_V$ . The space of  $V$ -valued tempered distributions we denote by  $\mathcal{S}'(\mathbf{R}^d, V)$  ( $f \in \mathcal{S}'(\mathbf{R}^d, V)$  is a continuous  $V$ -valued linear functional on  $\mathcal{S}(\mathbf{R}^d)$ ). If  $V = \mathbf{R}$ , we write  $\mathcal{S}'(\mathbf{R}^d, V) = \mathcal{S}'(\mathbf{R}^d)$  and denote by  $\langle \cdot, \cdot \rangle$  the duality between  $\mathcal{S}'(\mathbf{R}^d)$  and  $\mathcal{S}(\mathbf{R}^d)$ .

For a  $V$ -valued measurable function  $h$  on  $\mathbf{R}^d$  and  $p \geq 1$  we denote

$$|h|_{V,p}^p = \int_{\mathbf{R}^d} |h(x)|_V^p dx.$$

Further, for a characterization of our function spaces we will use the following construction (see [1]). By Lemma 6.1.7 in [1], there is a function  $\phi \in C_0^\infty(\mathbf{R}^d)$  such that  $\text{supp } \phi = \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ ,  $\phi(\xi) > 0$  if  $2^{-1} < |\xi| < 2$  and

$$\sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi) = 1 \quad \text{if } \xi \neq 0.$$

Define the functions  $\varphi_k \in \mathcal{S}(\mathbf{R}^d)$ ,  $k = 1, \dots$ , by

$$\mathcal{F}\varphi_k(\xi) = \phi(2^{-k}\xi),$$

and  $\varphi_0 \in \mathcal{S}(\mathbf{R}^d)$  by

$$\mathcal{F}\varphi_0(\xi) = 1 - \sum_{k \geq 1} \mathcal{F}\varphi_k(\xi).$$

Let  $\beta \in \mathbf{R}$  and  $p \geq 1$ . We introduce the Besov space  $B_{pp}^\beta(\mathbf{R}^d, V)$  of generalized functions  $f \in \mathcal{S}'(\mathbf{R}^d, V)$  with finite norm

$$|f|_{B_{pp}^\beta(\mathbf{R}^d, V)} = \left\{ \sum_{j=0}^{\infty} 2^{j\beta p} |\varphi_j * f|_{V,p}^p \right\}^{1/p}$$

and the Sobolev space  $H_p^\beta(\mathbf{R}^d, V)$  of  $f \in \mathcal{S}'(\mathbf{R}^d, V)$  with finite norm

$$|f|_{H_p^\beta(\mathbf{R}^d, V)} = |\mathcal{F}^{-1}((1 + |\xi|^2)^{\beta/2} \mathcal{F}f)|_{V,p} = \left| J^\beta f \right|_{V,p}, \quad (6)$$

where

$$J^\beta = (I - \Delta)^{\beta/2},$$

$I$  is the identity map and  $\Delta$  is the Laplacian in  $\mathbf{R}^d$ . For the scalar functions an equivalent norm to (6) is defined by

$$|f|_{H_p^\beta(\mathbf{R}^d)} = \left\{ \int_{\mathbf{R}^d} \left( \sum_{j=0}^{\infty} 2^{2\beta j} |\varphi_j * f(x)|^2 \right)^{p/2} dx \right\}^{1/p}. \quad (7)$$

We also introduce the corresponding spaces of generalized functions on  $E = [0, T] \times \mathbf{R}^d$ . The spaces  $B_{pp}^\beta(E, V)$  and  $H_p^\beta(E, V)$  consist of all measurable  $S'(\mathbf{R}^d, V)$ -valued functions  $f$  on  $[0, T]$  with finite norms

$$|f|_{B_{pp}^\beta(E, V)} = \left\{ \int_0^T |f(t)|_{B_{pp}^\beta(\mathbf{R}^d, V)}^p dt \right\}^{\frac{1}{p}}$$

and

$$|f|_{H_p^\beta(E, V)} = \left\{ \int_0^T |f(t)|_{H_p^\beta(\mathbf{R}^d, V)}^p dt \right\}^{\frac{1}{p}}.$$

Similarly we introduce the corresponding spaces of random generalized functions.

Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a complete probability space with a filtration of  $\sigma$ -algebras  $\mathbb{F} = (\mathcal{F}_t)$  satisfying the usual conditions. Let  $\mathcal{R}(\mathbb{F})$  be the progressive  $\sigma$ -algebra on  $[0, \infty) \times \Omega$ .

The spaces  $\mathbb{B}_{pp}^\beta(\mathbf{R}^d, V)$  and  $\mathbb{H}_p^\beta(\mathbf{R}^d, V)$  consist of all  $\mathcal{F}$ -measurable random functions  $f$  with values in  $B_{pp}^\beta(\mathbf{R}^d, V)$  and  $H_p^\beta(\mathbf{R}^d, V)$  with finite norms

$$|f|_{\mathbb{B}_{pp}^\beta(\mathbf{R}^d, V)} = \left\{ \mathbf{E} |f|_{B_{pp}^\beta(\mathbf{R}^d, V)}^p \right\}^{1/p}$$

and

$$|f|_{\mathbb{H}_p^\beta(\mathbf{R}^d, V)} = \left\{ \mathbf{E} |f|_{H_p^\beta(\mathbf{R}^d, V)}^p \right\}^{1/p}.$$

The spaces  $\mathbb{B}_{pp}^\beta(E, V)$  and  $\mathbb{H}_p^\beta(E, V)$  consist of all  $\mathcal{R}(\mathbb{F})$ -measurable random functions with values in  $B_{pp}^\beta(E, V)$  and  $H_p^\beta(E, V)$  with finite norms

$$|f|_{\mathbb{B}_{pp}^\beta(E, V)} = \left\{ \mathbf{E} |f|_{B_{pp}^\beta(E, V)}^p \right\}^{1/p}$$

and

$$|f|_{\mathbb{H}_p^\beta(E, V)} = \left\{ \mathbf{E} |f|_{H_p^\beta(E, V)}^p \right\}^{1/p}.$$

If  $V = L_r(U, \mathcal{U}, \Pi)$ ,  $r \geq 1$ , the space of  $r$ -integrable measurable functions on  $U$ , for brevity of notation we write

$$\begin{aligned} B_{r,pp}^\beta(A) &= B_{pp}^\beta(A, V), & \mathbb{B}_{r,pp}^\beta(A) &= \mathbb{B}_{pp}^\beta(A, V), \\ H_{r,p}^\beta(A) &= H_p^\beta(A, V), & \mathbb{H}_{r,p}^\beta(A) &= \mathbb{H}_p^\beta(A, V), \\ L_{r,p}(A) &= H_{r,p}^0(A), & \mathbb{L}_{r,p}(A) &= \mathbb{H}_{r,p}^0(A), \end{aligned}$$

where  $A = \mathbf{R}^d$  or  $E$ . For scalar functions we drop  $V$  in the notation of function spaces.

We also introduce the spaces  $\overline{\mathbb{B}}_{r,pp}^\beta(E)$  and  $\overline{\mathbb{H}}_{r,p}^\beta(E)$  consisting of  $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbf{R}_0^d)$ -measurable  $S'(\mathbf{R}^d)$ -valued random functions  $f = f(t, x, y)$  with finite norms

$$|f|_{\overline{\mathbb{B}}_{r,pp}^\beta(E)} = \left\{ \mathbf{E} \sum_{j=0}^{\infty} 2^{j\beta p} \int_0^T \int_{\mathbf{R}^d} \|(\varphi_j * f)(t, x, \cdot)\|_r^p dx dt \right\}^{1/p}$$

and

$$|f|_{\overline{\mathbb{H}}_{r,p}^\beta(E)} = \left\{ \mathbf{E} \int_0^T \int_{\mathbf{R}^d} \|J^\beta f(t, x, \cdot)\|_r^p dx dt \right\}^{1/p},$$

where

$$\|g(t, x, \cdot)\|_r = \left\{ \int_{\mathbf{R}_0^d} |g(t, x, y)|^r l^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}} \right\}^{1/r}.$$

### 2.3 Main results

We fix non-random functions  $m_0^{(\alpha)}(t, y) \geq 0$ ,  $\alpha \in (0, 2)$ , on  $[0, T] \times \mathbf{R}_0^d$  and positive constants  $K$  and  $\delta$ . Throughout the paper we assume that the functions  $m_0^{(\alpha)}$  satisfy the following conditions.

**Assumption A<sub>0</sub>.** (i) For each  $\alpha \in (0, 2)$  the function  $m_0^{(\alpha)}(t, y) \geq 0$  is measurable, homogeneous in  $y$  with index zero, differentiable in  $y$  up to the order  $d_0 = [\frac{d}{2}] + 2$  and

$$|D_y^\gamma m_0^{(\alpha)}(t, y)| \leq K$$

for all  $t \in [0, T]$ ,  $y \in \mathbf{R}_0^d$  and multiindices  $\gamma \in \mathbf{N}_0^d$  such that  $|\gamma| \leq d_0$ ;

(ii) For all  $t \in [0, T]$

$$\int_{S^{d-1}} w m_0^{(1)}(t, w) \mu_{d-1}(dw) = 0,$$



where  $S^{d-1}$  is the unit sphere in  $\mathbf{R}^d$  and  $\mu_{d-1}$  is the Lebesgue measure on it;

(iii) For each  $\alpha \in (0, 2)$  and  $t \in [0, T]$

$$\inf_{|\xi|=1} \int_{S^{d-1}} |(w, \xi)|^\alpha m_0^{(\alpha)}(t, w) \mu_{d-1}(dw) \geq \delta > 0.$$

**Remark 1** The nondegenerateness assumption  $A_0$  (iii) holds with certain  $\delta > 0$  if, e.g.

$$\inf_{t \in [0, T], w \in \Gamma} m_0^{(\alpha)}(t, w) > 0$$

for a measurable subset  $\Gamma \subset S^{d-1}$  of positive Lebesgue measure (the function  $m_0^{(\alpha)}$  can degenerate on a substantial set).

**Assumption A.** (i) The real-valued random functions  $m^{(\alpha)}(t, y)$  and  $l^{(\alpha)}(t, y)$  on  $[0, T] \times \mathbf{R}_0^d$  are non-negative and  $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbf{R}_0^d)$ -measurable; the real-valued random functions  $B^{ij}(t) = B^{ji}(t)$ ,  $b^i(t)$ ,  $i, j = 1, \dots, d$ , on  $[0, T]$  are  $\mathcal{R}(\mathbb{F})$ -measurable; the Hilbert space  $Y$ -valued random functions  $\sigma^i(t)$ ,  $i = 1, \dots, d$ , on  $[0, T]$  are  $\mathcal{R}(\mathbb{F})$ -measurable.

(ii) **P**-a.s. for all  $t \in [0, T]$ ,  $y \in \mathbf{R}_0^d$  and  $i, j = 1, \dots, d$

$$m^{(\alpha)}(t, y) + l^{(\alpha)}(t, y) + |B^{ij}(t)| + |b^i(t)| + |\sigma^i(t)|_Y \leq K$$

and for all  $0 < r < R < \infty$ ,

$$\int_{r \leq |y| \leq R} y m^{(1)}(t, y) \frac{dy}{|y|^{d+1}} = 0.$$

(iii) **P**-a.s. for all  $t \in [0, T]$  and  $y \in \mathbf{R}_0^d$

$$\begin{aligned} m^{(\alpha)}(t, y) - l^{(\alpha)}(t, y) &\geq m_0^{(\alpha)}(t, y) \quad \text{if } \alpha \in (0, 2), \\ (B^{ij}(t) - \frac{1}{2} \sigma^i(t) \cdot \sigma^j(t)) y_i y_j &\geq \delta |y|^2 \quad \text{if } \alpha = 2, \end{aligned}$$

where  $\delta > 0$ , the function  $m_0^{(\alpha)}$  satisfies Assumption  $A_0$  and  $\cdot$  denotes the inner product in  $Y$ .

**Remark 2** Assumption A (iii) is called **superparabolicity** of (1).

**Definition 3** Let  $\alpha \in (0, 2]$ ,  $\beta \in \mathbf{R}$ ,  $p \geq 2$ ,  $u_0 \in \mathbb{B}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(\mathbf{R}^d)$  be  $\mathcal{F}_0$ -measurable,  $f \in \mathbb{H}_p^\beta(E)$ ,  $\Phi \in \mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E) \cap \mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E)$ ,  $g \in \bar{\mathbb{B}}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E) \cap \bar{\mathbb{H}}_{2,p}^{\beta+\frac{\alpha}{2}}(E)$  and  $h \in \mathbb{H}_p^{\beta+\alpha/2}(E, Y)$ .

We say that  $u \in \mathbb{H}_p^{\beta+\alpha}(E)$  is a strong solution to (1) if  $u(t, \cdot)$  is strongly cadlag in  $H_p^\beta(\mathbf{R}^d)$  with respect to  $t$ ,  $A^{(\alpha)}u \in \mathbb{H}_p^\beta(E)$  and  $\mathbf{P}$ -a.s. in  $\varphi \in \mathcal{S}(\mathbf{R}^d)$

$$\begin{aligned} d\langle u(t), \varphi \rangle &= \left\langle A^{(\alpha)}u(t) - \lambda u(t) + f, \varphi \right\rangle dt \\ &\quad + \int_{\mathbf{R}_0^d} \langle u(t-, \cdot + y) - u(t-, \cdot) + g(t, \cdot, y), \varphi \rangle q^{(\alpha)}(dt, dy) 1_{\alpha \in (0, 2)} \\ &\quad + \int_U \langle \Phi(t, \cdot, v), \varphi \rangle \eta(dt, dv) + \langle 1_{\alpha=2} \sigma^i(t) \partial_i u(t) + h(t), \varphi \rangle dW_t, \\ u(0) &= u_0; \end{aligned} \tag{8}$$

equivalently, in integral form

$$\begin{aligned} \langle u(t), \varphi \rangle &= \langle u_0, \varphi \rangle + \int_0^t \left\langle A^{(\alpha)}u(s) - \lambda u(s) + f, \varphi \right\rangle ds \\ &\quad + \int_0^t \int_{\mathbf{R}_0^d} \langle u(s-, \cdot + y) - u(s-, \cdot) + g(s, \cdot, y), \varphi \rangle q^{(\alpha)}(ds, dy) 1_{\alpha \in (0, 2)} \\ &\quad + \int_0^t \int_U \langle \Phi(s, \cdot, v), \varphi \rangle \eta(ds, dv) + \int_0^t \langle 1_{\alpha=2} \sigma^i(s) \partial_i u(s) + h(s), \varphi \rangle dW_s, \end{aligned}$$

$$0 \leq t \leq T.$$

**Remark 4** Since according to Theorem 2.4.2 in [19]

$$\mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E) \cap \mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E) \subseteq \mathbb{H}_{p,p}^\beta(E) \cap \mathbb{H}_{2,p}^\beta(E), \bar{\mathbb{B}}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E) \cap \bar{\mathbb{H}}_{2,p}^{\beta+\frac{\alpha}{2}}(E) \subseteq \bar{\mathbb{H}}_{p,p}^\beta(E) \cap \bar{\mathbb{H}}_{2,p}^\beta(E),$$

the assumptions of Definition 3 imply (see Lemma 12 below) that

$$\begin{aligned} M_t^1 &= \int_0^t \int_{\mathbf{R}_0^d} g(s, \cdot, y) q^{(\alpha)}(ds, dy) 1_{\alpha \in (0, 2)}, \\ M_t^2 &= \int_0^t \int_U \Phi(s, \cdot, v) \eta(ds, dv), 0 \leq t \leq T, \end{aligned}$$

are cadlag  $H_p^\beta(\mathbf{R}^d)$ -valued. According to Theorem 1 in [16],

$$M_t^3 = \int_0^t [1_{\alpha=2} \sigma^i(s) \partial_i J^\beta u(s) + J^\beta h(s)] dW_s, 0 \leq t \leq T,$$

is continuous  $H_p^\beta(\mathbf{R}^d)$ -valued. By Corollary 13 below

$$Q(t) = \int_0^t \int [u(s, \cdot + y) - u(s, \cdot)] q^{(\alpha)}(ds, dy), 0 \leq t \leq T,$$

is  $H_p^\beta(\mathbf{R}^d)$ -valued cadlag.

The first main result concerns the so-called uncorrelated case of (1) defined by

$$\begin{aligned} du(t, x) &= (A^{(\alpha)}u - \lambda u + f)(t, x)dt \\ &\quad + \int_U \Phi(t, x, v)\eta(dt, dv) + h(t, x)dW_t \quad \text{in } E, \\ u(0, x) &= u_0(x) \quad \text{in } \mathbf{R}^d. \end{aligned} \tag{9}$$

**Theorem 5** *Let  $\alpha \in (0, 2]$ ,  $\beta \in \mathbf{R}$ ,  $p \geq 2$  and Assumption A be satisfied with  $l^{(\alpha)} = 0$  and  $\sigma^i = 0$ ,  $i = 1, \dots, d$ . Let  $u_0 \in \mathbb{B}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(\mathbf{R}^d)$  be  $\mathcal{F}_0$ -measurable,  $f \in \mathbb{H}_p^\beta(E)$ ,  $\Phi \in \mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E) \cap \mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E)$  and  $h \in \mathbb{H}_p^{\beta+\alpha/2}(E, Y)$ .*

*Then there is a unique strong solution  $u \in \mathbb{H}_p^{\beta+\alpha}(E)$  of (9). Moreover, there is a constant  $C = C(\alpha, \beta, p, d, T, K, \delta)$  such that*

$$\begin{aligned} |u|_{\mathbb{H}_p^{\beta+\alpha}(E)} &\leq C \left( |u_0|_{\mathbb{B}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} + |f|_{\mathbb{H}_p^\beta(E)} \right. \\ &\quad \left. + |\Phi|_{\mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E)} + |\Phi|_{\mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} + |h|_{\mathbb{H}_p^{\beta+\alpha/2}(E,Y)} \right). \end{aligned} \tag{10}$$

**Remark 6** According to the embedding theorem (see Theorem 6.4.4 in [1]), the estimate (10) holds with  $|u_0|_{\mathbb{B}_{pp}^\kappa(E)}$  and  $|\Phi|_{\mathbb{B}_{p,pp}^\kappa(E)}$  replaced by  $|u_0|_{\mathbb{H}_p^\kappa(E)}$  and  $|\Phi|_{\mathbb{H}_{p,p}^\kappa(E)}$ , where  $\kappa = \beta + \alpha - \frac{\alpha}{p}$ .

Theorem 5 covers the deterministic equation

$$\begin{aligned} \partial_t u(t, x) &= A^{(\alpha)}u(t, x) - \lambda u(t, x) + f(t, x) \quad \text{in } E, \\ u(0, x) &= u_0(x) \quad \text{in } \mathbf{R}^d \end{aligned} \tag{11}$$

with non-random coefficients and input functions. The following obvious consequence of Theorem 5 holds.

**Corollary 7** *Let  $\alpha \in (0, 2]$ ,  $\beta \in \mathbf{R}$ ,  $p \geq 2$  and for all  $t \in [0, T]$ ,  $y \in \mathbf{R}_0^d$ ,*

$$\begin{aligned} m^{(\alpha)}(t, y) + |B^{ij}(t)| + |b^i(t)| &\leq K, \quad i, j = 1, \dots, d, \\ m^{(\alpha)}(t, y) &\geq m_0^{(\alpha)}(t, y) \quad \text{if } \alpha \in (0, 2) \end{aligned}$$

and

$$B^{ij}(t)y_i y_j \geq \delta |y|^2 \quad \text{if } \alpha = 2,$$

where  $m_0^{(\alpha)}$  satisfies Assumption A<sub>0</sub>. Let  $u_0 \in B_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(\mathbf{R}^d)$  and  $f \in H_p^\beta(E)$ .

Then there is a unique strong solution  $u \in H_p^{\beta+\alpha}(E)$  of (11). Moreover, there is a constant  $C = C(\alpha, \beta, p, d, T, K, \delta)$  such that

$$|u|_{H_p^{\beta+\alpha}(E)} \leq C \left( |u_0|_{B_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} + |f|_{H_p^\beta(E)} \right).$$

Given  $g \in \bar{\mathbb{B}}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E) \cap \bar{\mathbb{H}}_{2,p}^{\beta+\frac{\alpha}{2}}(E)$  we denote

$$\Lambda g(t, x, y) = g(t, x - y, y), (t, x) \in E, y \in \mathbf{R}_0^d.$$

For a  $g, \Lambda g \in \bar{\mathbb{B}}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E) \cap \bar{\mathbb{H}}_{2,p}^{\beta+\frac{\alpha}{2}}(E)$  we denote

$$Ig(t, x) = \int_{\mathbf{R}_0^d} (\Lambda g - g)(t, x, y) l^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}}(t, x) \in E,$$

assuming that

$$Ig(t, x) = \lim_{\varepsilon \rightarrow 0} \int_{|y| > \varepsilon} (\Lambda g - g)(t, x, y) l^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}}(t, x) \in E, \quad (12)$$

is well defined as a limit in  $\mathbb{H}_p^\beta(E)$  (we write simply that  $Ig \in \mathbb{H}_p^\beta(E)$  in this case).

In the general case the following statement holds for (1).

**Theorem 8** Let  $\alpha \in (0, 2]$ ,  $\beta \in \mathbf{R}$ ,  $p \geq 2$  and Assumption A be satisfied.

Let  $u_0 \in \bar{\mathbb{B}}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(\mathbf{R}^d)$  be  $\mathcal{F}_0$ -measurable,  $f, Ig \in \mathbb{H}_p^\beta(E)$ ,  $\Phi \in \bar{\mathbb{B}}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E) \cap \bar{\mathbb{H}}_{2,p}^{\beta+\frac{\alpha}{2}}(E)$ ,  $g, \Lambda g \in \bar{\mathbb{B}}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E) \cap \bar{\mathbb{H}}_{2,p}^{\beta+\frac{\alpha}{2}}(E)$  and  $h \in \mathbb{H}_p^{\beta+\alpha/2}(E, Y)$ .

Then there is a unique strong solution  $u \in \mathbb{H}_p^{\beta+\alpha}(E)$  of (1). Moreover, there is a constant  $C = C(\alpha, \beta, p, d, T, K, \delta)$  such that

$$\begin{aligned} |u|_{\mathbb{H}_p^{\beta+\alpha}(E)} \leq & C \left( |u_0|_{\bar{\mathbb{B}}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} + |f - Ig|_{\mathbb{H}_p^\beta(E)} \right. \\ & + |h|_{\mathbb{H}_p^{\beta+\alpha/2}(E, Y)} + |\Phi|_{\bar{\mathbb{H}}_{2,p}^{\beta+\frac{\alpha}{2}}(E)} + |\Phi|_{\bar{\mathbb{B}}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} \\ & \left. + |\Lambda g|_{\bar{\mathbb{H}}_{2,p}^{\beta+\frac{\alpha}{2}}(E)} + |\Lambda g|_{\bar{\mathbb{B}}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} \right). \end{aligned}$$

### 3 Approximation of input functions

Let a non-negative function  $\zeta \in C_0^\infty(\mathbf{R}^d)$  be such that  $\zeta(x) = 0$  if  $|x| \geq 1$  and  $\int \zeta(x)dx = 1$ . For  $\varepsilon \in (0, 1)$  we set

$$\zeta_\varepsilon(x) = \varepsilon^{-d} \zeta(x/\varepsilon), \quad x \in \mathbf{R}^d.$$

Let  $V$  be a Banach space with a norm  $|\cdot|_V$ .

**Lemma 9** *Let  $\beta \in \mathbf{R}$ ,  $p \geq 1$  and  $u \in A$ , where  $A = B_{pp}^\beta(\mathbf{R}^d, V)$  or  $H_p^\beta(\mathbf{R}^d, V)$ . Let  $u_\varepsilon = u * \zeta_\varepsilon$ .*

*Then  $|u_\varepsilon - u|_A \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Moreover, for every  $\varepsilon$  and multiindex  $\gamma \in \mathbf{N}_0^d$  there is a constant  $C$  not depending on  $u$  such that*

$$\sup_x |\partial^\gamma u_\varepsilon(x)|_V + |\partial^\gamma u_\varepsilon|_{V,p} \leq C|u|_A.$$

**Proof.** Let  $v \in L_p(\mathbf{R}^d, V)$ ,  $p \geq 1$ , and  $v_\varepsilon = v * \zeta_\varepsilon$ . It is well known that

$$|v_\varepsilon|_{V,p} \leq |v|_{V,p}$$

and

$$|v - v_\varepsilon|_{V,p} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

Therefore,

$$\begin{aligned} |u_\varepsilon|_{B_{pp}^\beta(\mathbf{R}^d, V)}^p &= \sum_{j=0}^{\infty} 2^{j\beta p} |\varphi_j * u_\varepsilon|_{V,p}^p = \sum_{j=0}^{\infty} 2^{j\beta p} |(\varphi_j * u)_\varepsilon|_{V,p}^p \\ &\leq \sum_{j=0}^{\infty} 2^{j\beta p} |\varphi_j * u|_{V,p}^p = |u|_{B_{pp}^\beta(\mathbf{R}^d, V)}^p \end{aligned}$$

and

$$\begin{aligned} |u_\varepsilon - u|_{B_{pp}^\beta(\mathbf{R}^d, V)}^p &= \sum_{j=0}^{\infty} 2^{j\beta p} |\varphi_j * (u_\varepsilon - u)|_{V,p}^p \\ &= \sum_{j=0}^{\infty} 2^{j\beta p} |(\varphi_j * u)_\varepsilon - \varphi_j * u|_{V,p}^p \rightarrow 0 \end{aligned}$$

as  $\varepsilon \rightarrow 0$ . Similarly,

$$|u_\varepsilon|_{H_p^\beta(\mathbf{R}^d, V)} = |J^\beta u_\varepsilon|_{V,p} = |(J^\beta u)_\varepsilon|_{V,p} \leq |J^\beta u|_{V,p} = |u|_{H_p^\beta(\mathbf{R}^d, V)}$$

and

$$|u_\varepsilon - u|_{H_p^\beta(\mathbf{R}^d, V)} = |J^\beta(u_\varepsilon - u)|_{V, p} = \left| \left( J^\beta u \right)_\varepsilon - J^\beta u \right|_{V, p} \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ .

Let  $u \in B_{pp}^\beta(\mathbf{R}^d, V)$ . Then

$$u = \sum_{j=0}^{\infty} u * \varphi_j$$

in  $\mathcal{S}'(\mathbf{R}^d)$ . Therefore,

$$u_\varepsilon = \sum_{j=0}^{\infty} u * \varphi_j * \zeta_\varepsilon$$

and, for every  $m \in \mathbf{N}_0$  and  $l > 0$ ,

$$J^m u_\varepsilon = \sum_{j=0}^{\infty} J^{-l} u * \varphi_j * J^{m+l} \zeta_\varepsilon.$$

Applying Minkowski's and Hölder's inequalities, we get

$$\begin{aligned} |J^m u_\varepsilon(x)|_V &\leq \sum_{j=0}^{\infty} |J^{-l} u * \varphi_j * J^{m+l} \zeta_\varepsilon|_V \leq \sum_{j=0}^{\infty} |J^{-l} u * \varphi_j|_V * |J^{m+l} \zeta_\varepsilon|(x) \\ &\leq C \sum_{j=0}^{\infty} |J^{-l} u * \varphi_j|_{V, p} \end{aligned}$$

and

$$|J^m u_\varepsilon|_{V, p} \leq \sum_{j=0}^{\infty} |J^{-l} u * \varphi_j * J^{m+l} \zeta_\varepsilon|_{V, p} \leq C \sum_{j=0}^{\infty} |J^{-l} u * \varphi_j|_{V, p}.$$

By Lemma 6.2.1 in [1],

$$|J^{-l} u * \varphi_j|_{V, p} \leq C 2^{-lj} |u * \varphi_j|_{V, p}.$$

On the other hand,

$$|u * \varphi_j|_{V, p} \leq 2^{-\beta j} |u|_{B_{pp}^\beta(\mathbf{R}^d, V)}, \quad j \geq 0.$$

Choosing  $l$  so that  $l + \beta > 0$ , we have

$$\begin{aligned} |J^m u_\varepsilon(x)|_V + |J^m u_\varepsilon|_{V,p} &\leq C \sum_{j=0}^{\infty} |J^{-l} u * \varphi_j|_{V,p} \leq C \sum_{j=0}^{\infty} 2^{-lj} |u * \varphi_j|_{V,p} \\ &\leq C \sum_{j=0}^{\infty} 2^{-(\beta+l)j} |u|_{B_{pp}^\beta(\mathbf{R}^d, V)} \leq C |u|_{B_{pp}^\beta(\mathbf{R}^d, V)}. \end{aligned}$$

For  $u \in H_p^\beta(\mathbf{R}^d, V)$ , we have

$$J^m u_\varepsilon = J^\beta u * J^{m-\beta} \zeta_\varepsilon.$$

Applying Minkowski's and Hölder's inequalities, we get

$$|J^m u_\varepsilon(x)|_V + |J^m u_\varepsilon|_{V,p} \leq C |J^\beta u|_{V,p} = C |u|_{H_p^\beta(\mathbf{R}^d, V)}.$$

The lemma is proved.  $\blacksquare$

Similarly we approximate random functions.

**Lemma 10** *Let  $\beta \in \mathbf{R}$ ,  $p \geq 1$  and  $g \in \mathbb{A}$ , where  $\mathbb{A} = \mathbb{B}_{pp}^\beta(E, V)$  or  $\mathbb{H}_p^\beta(E, V)$ . Let*

$$(g)_n = g_n(t, x) = n \int_{t_n}^t g(s, \cdot) * \zeta_{\frac{1}{n}}(x) ds, \quad (t, x) \in E, \quad (13)$$

where  $t_n = (t - \frac{1}{n}) \vee 0$ .

Then  $|g_n - g|_{\mathbb{A}} \rightarrow 0$  as  $n \rightarrow \infty$ . Moreover, for every  $n$  and multiindex  $\gamma \in \mathbf{N}_0^d$  there is a constant  $C$  not depending on  $u$  such that

$$\begin{aligned} &\mathbf{E} \left[ \sup_{(t,x) \in E} |\partial_x^\gamma g_n(t, x)|_V^p + \int_0^T |\partial_x^\gamma g_n(t, \cdot)|_{V,p}^p dt \right] \\ &\leq nC |g|_{\mathbb{A}} < \infty. \end{aligned}$$

**Proof.** Let

$$\tilde{g}_n(t, x) = n \int_{t_n}^t g(s, x) ds, \quad \bar{g}_n(t, x) = g(t, \cdot) * \zeta_{\frac{1}{n}}(x).$$

Applying Minkowski's and Hölder's inequalities, we have

$$\begin{aligned} |g_n - \tilde{g}_n|_{\mathbb{A}}^p &= \int_0^T \left| n \int_{t_n}^t [\bar{g}_n(s, \cdot) - g(s, \cdot)] ds \right|_{\mathbb{A}_d}^p dt \\ &\leq \int_0^T n \int_{t_n}^t |\bar{g}_n(s, \cdot) - g(s, \cdot)|_{\mathbb{A}_d}^p ds dt \\ &\leq \int_0^T |\bar{g}_n(s, \cdot) - g(s, \cdot)|_{\mathbb{A}_d}^p ds, \end{aligned}$$

where  $\mathbb{A}_d = \mathbb{B}_{pp}^\beta(\mathbf{R}^d, V)$  or  $\mathbb{H}_p^\beta(\mathbf{R}^d, V)$ . Hence, by Lemma 9,  $|g_n - \tilde{g}_n|_{\mathbb{A}} \rightarrow 0$  as  $n \rightarrow \infty$ .

Let  $v(t)$ ,  $t \in [0, T]$ , be a function in a Banach space with norm  $\|\cdot\|$  such that  $\int_0^T \|v(t)\|^p dt < \infty$ . It is well known that

$$n \int_0^T \int_{t_n}^t \|v(s) - v(t)\|^p ds dt \rightarrow 0$$

as  $n \rightarrow \infty$ . Therefore, applying Minkowski's and Hölder's inequalities, we get

$$\begin{aligned} |\tilde{g}_n - g|_{\mathbb{A}}^p &= \int_0^T \left| n \int_{t_n}^t [g(s, \cdot) - g(t, \cdot)] ds \right|_{\mathbb{A}_d}^p dt \\ &\leq \int_0^T \left[ n \int_{t_n}^t |g(s, \cdot) - g(t, \cdot)|_{\mathbb{A}_d} ds \right]^p dt \\ &\leq n \int_0^T \int_{t_n}^t |g(s, \cdot) - g(t, \cdot)|_{\mathbb{A}_d}^p ds dt \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . Hence,

$$|g_n - g|_{\mathbb{A}} \leq |g_n - \tilde{g}_n|_{\mathbb{A}} + |\tilde{g}_n - g|_{\mathbb{A}} \rightarrow 0, \quad n \rightarrow \infty.$$

Applying Minkowski's and Hölder's inequalities, we have

$$\begin{aligned} |\partial_x^\gamma g_n(t, x)|_V^p &= \left| n \int_{t_n}^t \partial_x^\gamma \bar{g}_n(s, x) ds \right|_V^p \leq \left( n \int_{t_n}^t |\partial_x^\gamma \bar{g}_n(s, x)|_V ds \right)^p \\ &\leq n \int_{t_n}^t |\partial_x^\gamma \bar{g}_n(s, x)|_V^p ds. \end{aligned}$$

Therefore, by Lemma 9,

$$\begin{aligned} \mathbf{E} &\left( \sup_{(t,x) \in E} |\partial_x^\gamma g_n(t, x)|_V^p + \int_0^T |\partial_x^\gamma g_n(t, \cdot)|_{V,p}^p dt \right) \\ &\leq n \mathbf{E} \left( \sup_{(t,x) \in E} \int_{t_n}^t |\partial_x^\gamma \bar{g}_n(s, x)|_V^p ds + \int_0^T \int_{t_n}^t |\partial_x^\gamma \bar{g}_n(s, \cdot)|_{V,p}^p ds dt \right) \\ &\leq nC |g|_{\mathbb{A}}^p < \infty. \end{aligned}$$

The lemma is proved. ■

We denote by  $\mathfrak{D}_p(E, V)$ ,  $p \geq 1$ , the space of all  $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbf{R}^d)$ -measurable  $V$ -valued random functions  $\Phi$  on  $E$  such that  $\Phi \in \cap_{\kappa > 0} \mathbb{H}_p^\kappa(E, V)$  and for every multiindex  $\gamma \in \mathbf{N}_0^d$

$$\mathbf{E} \sup_{(t,x) \in E} |D_x^\gamma \Phi(t, x)|_V^p < \infty.$$



Similarly we define the space  $\mathfrak{D}_p(\mathbf{R}^d, V)$  replacing  $\mathcal{R}(\mathbb{F})$  and  $E$  by  $\mathcal{F}$  and  $\mathbf{R}^d$  in the definition of  $\mathfrak{D}_p(E, V)$ . For brevity of notation, if  $V = L_r(U, \mathcal{U}, \Pi)$ ,  $r \geq 1$ , we write  $\mathfrak{D}_{r,p}(E) = \mathfrak{D}_p(E, V)$ . If  $V = \mathbf{R}$ , we drop  $V$  in  $\mathfrak{D}_p(E, V)$ .

We denote by  $\bar{\mathfrak{D}}_{r,p}(E)$ ,  $r \geq 1$ ,  $p \geq 1$ , the space of all  $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{B}(\mathbf{R}_0^d)$ -measurable real-valued random functions  $g$  such that for every multiindex  $\gamma \in \mathbf{N}_0^d$

$$\mathbf{E} \left\{ \sup_{(t,x) \in E} \left[ \int_{\mathbf{R}_0^d} |D_x^\gamma g(t, x, y)|^r l^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}} \right]^{p/r} + \int_0^T \int_{\mathbf{R}^d} \left[ \int_{\mathbf{R}_0^d} |D_x^\gamma g(t, x, y)|^r l^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}} \right]^{p/r} dx dt \right\} < \infty.$$

Lemmas 9 and 10 imply the following statement.

**Lemma 11** *Let  $p \geq 1$ ,  $r \geq 1$  and  $\kappa, \kappa' \in \mathbf{R}$ . Then:*

- (a) *the set  $\mathfrak{D}_p(\mathbf{R}^d)$  is a dense subset in  $\mathbb{B}_{pp}^\kappa(\mathbf{R}^d)$  and the set  $\mathfrak{D}_p(E)$  is a dense subset of  $\mathbb{H}_p^\kappa(E)$ ;*
- (b) *the set  $\mathfrak{D}_{r,p}(E)$  is a dense subset in  $\mathbb{H}_{r,p}^\kappa(E)$  and  $\mathbb{B}_{r,pp}^\kappa(E)$ , and the set  $\mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E)$  is a dense subset of  $\mathbb{H}_{2,p}^\kappa(E) \cap \mathbb{B}_{p,pp}^{\kappa'}(E)$ ;*
- (c) *the set  $\bar{\mathfrak{D}}_{r,p}(E)$  is a dense subset in  $\mathbb{H}_{r,p}^\kappa(E)$  and  $\mathbb{B}_{r,pp}^\kappa(E)$ , and the set  $\bar{\mathfrak{D}}_{2,p}(E) \cap \bar{\mathfrak{D}}_{p,p}(E)$  is a dense subset of  $\mathbb{H}_{2,p}^\kappa(E) \cap \mathbb{B}_{p,pp}^{\kappa'}(E)$ .*

**Proof.** (a) Let  $u \in \mathbb{B}_{pp}^\kappa(\mathbf{R}^d)$  and  $u_n(x) = u * \zeta_{1/n}(x)$ ,  $n = 1, 2, \dots$ . Using Lemma 9, it is easy to derive that  $u \in \mathfrak{D}_p(\mathbf{R}^d)$  and  $|u - u_n|_{\mathbb{B}_{pp}^\kappa(\mathbf{R}^d)} \rightarrow 0$  as  $n \rightarrow \infty$ . If  $g \in \mathbb{H}_p^\kappa(E)$  and  $g_n$  is defined by (13), then by Lemma 10,  $|g - g_n|_{\mathbb{H}_p^\kappa(E)} \rightarrow 0$  as  $n \rightarrow \infty$ .

(b) According to Lemma 10, we have the following statements:

- (i) if  $g \in \mathbb{H}_{r,p}^\kappa(E)$  or  $\mathbb{B}_{r,pp}^\kappa(E)$ , then the functions  $g_n$  defined by (13) belong to  $\mathfrak{D}_{r,p}(E)$  and  $|g - g_n|_{\mathbb{H}_{r,p}^\kappa(E)} \rightarrow 0$  or  $|g - g_n|_{\mathbb{B}_{r,pp}^\kappa(E)}$  as  $n \rightarrow \infty$ ;
- (ii) if  $g \in \mathbb{H}_{2,p}^\kappa(E) \cap \mathbb{B}_{p,pp}^{\kappa'}(E)$ , then  $g_n \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E)$ ,  $n = 1, \dots$ , and  $|g - g_n|_{\mathbb{H}_{2,p}^\kappa(E)} + |g - g_n|_{\mathbb{B}_{p,pp}^{\kappa'}(E)} \rightarrow 0$  as  $n \rightarrow \infty$ .

(c) According to Lemma 14.50 in [7], there is a  $\mathbf{R}^d$ -valued  $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbf{R}_0)$ -measurable function  $c^{(\alpha)}(t, z)$  such that

$$l^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}} = \int_{\mathbf{R}_0} 1_{dy}(c^{(\alpha)}(t, z)) \frac{dz}{z^2} \quad (14)$$

and for any non-negative measurable function  $F(t, x, y)$

$$\int_{\mathbf{R}_0^d} F(t, x, y) l^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}} = \int_{\mathbf{R}_0} \tilde{F}(t, x, z) \frac{dz}{z^2},$$

where  $\tilde{F}(t, x, z) = F(t, x, c^{(\alpha)}(t, z))$ . Hence, if  $F \in \bar{\mathbb{H}}_{r,p}^\kappa(E)$ , then  $|F|_{\bar{\mathbb{H}}_{r,p}^\kappa(E)} = |\tilde{F}|_{\mathbb{H}_p^\kappa(E, V_r)}$ , where  $V_r = L_r(\mathbf{R}_0, \mathcal{B}(\mathbf{R}_0), dz/z^2)$ . Also, if  $F \in \bar{\mathbb{B}}_{r,pp}^\kappa(E)$ , then  $|F|_{\bar{\mathbb{B}}_{r,pp}^\kappa(E)} = |\tilde{F}|_{\mathbb{B}_{pp}^\kappa(E, V_r)}$ .

Let  $g \in \bar{\mathbb{H}}_{r,p}^\kappa(E)$  and  $g_n$  be the function defined by (13). By Lemma 10,  $\tilde{g}_n \in \mathfrak{D}_p(E, V_r)$  and  $|\tilde{g} - \tilde{g}_n|_{\mathbb{H}_p^\kappa(E, V_r)} \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore,  $g_n \in \bar{\mathfrak{D}}_{r,p}(E)$  and  $|g - g_n|_{\bar{\mathbb{H}}_{r,p}^\kappa(E)} = |\tilde{g} - \tilde{g}_n|_{\mathbb{H}_p^\kappa(E, V_r)} \rightarrow 0$  as  $n \rightarrow \infty$ . So,  $\bar{\mathfrak{D}}_{r,p}(E)$  is dense in  $\bar{\mathbb{H}}_{r,p}^\kappa(E)$ .

Similarly we prove the remaining assertions of part (c). ■

### 3.1 Stochastic integrals

We discuss here the definition of the stochastic integrals with respect to a martingale measure  $\eta$ .

**Lemma 12** *Let  $\beta \in \mathbf{R}, p \geq 2, \Phi \in \mathbb{H}_{2,p}^\beta(E) \cap \mathbb{H}_{p,p}^\beta(E)$ . There is a unique cadlag  $H_p^\beta(\mathbf{R}^d)$ -valued process*

$$M(t) = \int_0^t \int \Phi(s, x, v) \eta(ds, dv), 0 \leq t \leq T, x \in \mathbf{R}^d,$$

such that for every  $\varphi \in \mathcal{S}(\mathbf{R}^d)$

$$\langle M(t), \varphi \rangle = \int_0^t \int \langle \Phi(s, \cdot, v), \varphi \rangle \eta(ds, dv), 0 \leq t \leq T. \quad (15)$$

Moreover, there is a constant independent of  $C$  such that

$$\mathbf{E} \sup_{t \leq T} \left| \int_0^t \int \Phi(s, \cdot, v) \eta(ds, dv) \right|_{H_p^\beta(\mathbf{R}^d)} \leq C \sum_{r=2,p} |\Phi|_{\mathbb{H}_{r,p}^\beta(E)}.$$

**Proof.** For an arbitrary  $\phi \in \mathbb{H}_{2,p}^\beta(E) \cap \mathbb{H}_{p,p}^\beta(E)$ , by stochastic Fubini theorem (Lemma 2 in [15])

$$\begin{aligned} \int_0^t \int \langle \phi(s, \cdot, v), \varphi \rangle \eta(ds, dv) &= \int_0^t \int \int J^\beta \phi(s, x, v) J^{-\beta} \varphi(x) dx \eta(ds, dv) \\ &= \int \int_0^t \int J^\beta \phi(s, x, v) \eta(ds, dv) J^{-\beta} \varphi(x) dx, \end{aligned}$$

and (see Corollary 2 in [13])

$$\begin{aligned} & \mathbf{E} \left[ \sup_{t \leq T} \left| \int_0^t \int \langle \phi(s, \cdot, v), \varphi \rangle \eta(ds, dv) \right|^p \right] \\ & \leq C \int \mathbf{E} \sup_{t \leq T} \left| \int_0^t \int J^\beta \phi(s, x, v) \eta(ds, dv) \right|^p dx \leq C \sum_{r=2,p} |\phi|_{\mathbb{H}_{r,p}^\beta(E)}. \end{aligned} \quad (16)$$

First we define the stochastic integral for  $\Phi \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E)$ . By Lemma 15 in [15], for a given  $\Phi \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E)$  there is a cadlag in  $t$  and smooth in  $x$  adapted function  $M(t, x)$  such that for each  $\gamma \in \mathbf{N}_0^d$  and  $x \in \mathbf{R}^d$ ,  $\mathbf{P}$ -a.s.

$$D_x^\gamma M(t, x) = \int_0^t \int D_x^\gamma \Phi(s, x, v) \eta(ds, dv), 0 \leq t \leq T.$$

By stochastic Fubini theorem (Lemma 2 in [15]), for each  $\beta \in \mathbf{R}$  and  $x \in \mathbf{R}^d$ ,  $\mathbf{P}$ -a.s.

$$J^\beta M(t, x) = \int_0^t \int J^\beta \Phi(s, x, v) \eta(ds, dv), 0 \leq t \leq T,$$

and  $\mathbf{P}$ -a.s

$$\begin{aligned} \langle M(t), \varphi \rangle &= \int M(t, x) \varphi(x) dx = \int_0^t \int \left( \int \Phi(s, x, v) \varphi(x) dx \right) \eta(ds, dv) \\ &= \int_0^t \int \langle \Phi(s, \cdot, v), \varphi \rangle \eta(ds, dv), 0 \leq t \leq T. \end{aligned} \quad (17)$$

Obviously,  $J^\beta M(t)$  is  $\mathbb{L}_p(\mathbf{R}^d)$ -valued continuous and, by Corollary 2 in [13], there is a constant independent of  $\Phi$  such that

$$\mathbf{E} \sup_{t \leq T} |M(t)|_{H_p^\beta(\mathbf{R}^d)}^p \leq C \sum_{r=2,p} |\Phi|_{\mathbb{H}_{r,p}^\beta(E)}^p. \quad (18)$$

If  $\Phi \in \mathbb{H}_{2,p}^\beta(E) \cap \mathbb{H}_{p,p}^\beta(E)$ , then there is a sequence  $\Phi_n \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E)$  such that

$$\sum_{r=2,p} |\Phi - \Phi_n|_{\mathbb{H}_{r,p}^\beta(E)} \rightarrow 0$$

as  $n \rightarrow \infty$ . Let

$$M_n(t) = \int_0^t \int \Phi_n(t, \cdot, v) \eta(ds, dv), 0 \leq t \leq T.$$

According to (18) and (17),

$$\mathbf{E} \sup_{t \leq T} |M_n(t) - M_m(t)|_{H_p^\beta(\mathbf{R}^d)}^p \leq C \sum_{r=2,p} |\Phi_n - \Phi_m|_{\mathbb{H}_{r,p}^\beta(E)}^p \rightarrow 0$$

as  $n, m \rightarrow \infty$ . Therefore there is an adapted cadlag  $H_p^\beta(\mathbf{R}^d)$ -valued process  $M(t)$  so that

$$\mathbf{E} \sup_{t \leq T} |M_n(t) - M(t)|_{H_p^\beta(\mathbf{R}^d)}^p \rightarrow 0$$

as  $n \rightarrow \infty$ . On the other hand by (16),

$$\begin{aligned} & \mathbf{E} \sup_{t \leq T} \left| \int_0^t \int \langle \Phi_n(s, \cdot, v) - \Phi(s, \cdot, v), \varphi \rangle \eta(ds, dv) \right|^p \\ & \leq C \sum_{r=2,p} |\Phi_n - \Phi|_{\mathbb{H}_{r,p}^\beta(E)}^p \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ , and (15) holds. The statement follows. ■

**Corollary 13** *Let  $\alpha \in (0, 2)$ ,  $\beta \in \mathbf{R}$ ,  $p \geq 2$ ,  $u \in \mathbb{H}_p^{\beta+\alpha}(E)$ . Then*

$$Q(t) = \int_0^t \int [u(s, \cdot + y) - u(s, \cdot)] q^{(\alpha)}(ds, dy), 0 \leq t \leq T,$$

*is cadlag  $H_p^\beta(\mathbf{R}^d)$ -valued and*

$$\mathbf{E} \sup_{t \leq T} |Q(t)|_{H_p^\beta(\mathbf{R}^d)} \leq C |u|_{\mathbb{H}_p^{\beta+\alpha/2}(E)}.$$

**Proof.** We apply Lemma 12 with  $\Phi(s, x, y) = u(s, x + y) - u(s, x)$ ,  $(s, x) \in E, y \in \mathbf{R}_0^d$ . We have

$$J^\beta \Phi(s, x, y) = J^\beta u(s, x + y) - J^\beta u(s, x)$$

and, by Theorem 2.2 in [18],

$$\begin{aligned} & \int \left( \int |J^\beta \Phi(s, x, y)|^2 \frac{dy}{|y|^{d+\alpha}} \right)^{p/2} dx \\ & = \int \left( \int |J^\beta u(s, x + y) - J^\beta u(s, x)|^2 \frac{dy}{|y|^{d+\alpha}} \right)^{p/2} \\ & \leq C |u|_{\mathbb{H}_p^{\beta+\alpha/2}}^p. \end{aligned}$$

By definition of the norm,

$$\begin{aligned}
& \mathbf{E} \int_0^T \int \int |J^\beta \Phi(s, x, y)|^p \frac{dy dx ds}{|y|^{d+\alpha}} \\
&= \mathbf{E} \int_0^T \int \int |J^\beta u(s, x+y) - J^\beta u(s, x)|^p \frac{dy dx ds}{|y|^{d+\alpha}} \\
&\leq C |u|_{\mathbb{B}_{pp}^{\beta+\alpha/p}}^p \leq C |u|_{\mathbb{H}_p^{\beta+\alpha/2}}^p.
\end{aligned}$$

■

## 4 Model problem. Partial case I

In this section, we consider the Cauchy problem

$$\begin{aligned}
du(t, x) &= (A_0^{(\alpha)} u - \lambda u + f)(t, x) dt \\
&\quad + \int_U \Phi(t, x, v) \eta(dt, dv) + h(t, x) dW_t \quad \text{in } E, \\
u(0, x) &= u_0(x) \quad \text{in } \mathbf{R}^d
\end{aligned} \tag{19}$$

for smooth in  $x$  input functions  $u_0, f, \Phi, h$ .

The operator  $A_0^{(\alpha)}$ ,  $\alpha \in (0, 2]$ , is defined by

$$\begin{aligned}
A_0^{(\alpha)} u(t, x) &= \int \nabla_y^\alpha u(t, x) m_0^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}} 1_{\alpha \in (0, 2)} \\
&\quad + (b(t), \nabla u(t, x)) 1_{\alpha=1} + \frac{1}{2} \delta \Delta u(t, x) 1_{\alpha=2},
\end{aligned}$$

where  $\delta > 0$ ,

$$\nabla_y^\alpha u(t, x) = u(t, x+y) - u(t, x) - (\nabla u(t, x), y) \chi^{(\alpha)}(y),$$

$\chi^{(\alpha)}(y) = 1_{\alpha \in (1, 2)} + 1_{|y| \leq 1} 1_{\alpha=1}$  and  $\Delta$  is the Laplace operator in  $\mathbf{R}^d$ .

### 4.1 Auxiliary results

In terms of Fourier transform

$$\mathcal{F}(A_0^{(\alpha)} u)(t, \xi) = \psi_0^{(\alpha)}(t, \xi) \mathcal{F}u(t, \xi),$$

where

$$\begin{aligned}\psi_0^{(\alpha)}(t, \xi) = & -c_0 \int_{S^{d-1}} |(w, \xi)|^\alpha \left[ 1 - i \left( \operatorname{tg} \frac{\alpha\pi}{2} \operatorname{sgn}(w, \xi) 1_{\alpha \neq 1} \right. \right. \\ & \left. \left. - \frac{2}{\pi} \operatorname{sgn}(w, \xi) \ln |(w, \xi)| 1_{\alpha=1} \right) \right] m_0^{(\alpha)}(t, w) \mu_{d-1}(dw) 1_{\alpha \in (1, 2)} \\ & + i(b(t), \xi) 1_{\alpha=1} - \frac{1}{2} \delta |\xi|^2 1_{\alpha=2}.\end{aligned}$$

and  $c_0 = c_0(\alpha)$  is a positive constant.

Let us introduce the functions

$$\begin{aligned}G_{s,t}^{(\alpha)}(x) &= \mathcal{F}^{-1} \left\{ \exp \left( \int_s^t \psi_0^{(\alpha)}(r, \xi) dr \right) \right\} (x), \\ G_{s,t}^{(\alpha), \lambda}(x) &= e^{-\lambda(t-s)} G_{s,t}^{(\alpha)}(x) \quad 0 \leq s < t \leq T, \quad x \in \mathbf{R}^d.\end{aligned}$$

**Remark 14** The function  $G_{s,t}^{(\alpha)}(x)$  is the fundamental solution of the equation  $\partial_t u = A_0^{(\alpha)} u$ . On the other hand (see, e.g. [17]),  $G_{s,t}^{(\alpha)}$  is the density function of an  $\alpha$ -stable distribution. Hence,  $G_{s,t}^{(\alpha)} \geq 0$  and

$$\int_{\mathbf{R}^d} G_{s,t}^{(\alpha)}(x) dx = 1.$$

Further, for brevity of notation, we will drop the superscript  $\alpha$  in  $G_{s,t}^{(\alpha), \lambda}$  and  $G_{s,t}^{(\alpha)}$ .

For a representation of solution to (19) we introduce the following operators:

$$\begin{aligned}T_t^\lambda u_0(x) &= G_{0,t}^\lambda * u_0(x), \quad u_0 \in \mathfrak{D}_p(\mathbf{R}^d), \\ R_\lambda f(t, x) &= \int_0^t G_{s,t}^\lambda * f(s, x) ds, \quad f \in \mathfrak{D}_p(E), \\ \tilde{R}_\lambda \Phi(t, x) &= \int_0^t \int_U G_{s,t}^\lambda * \Phi(s, x, v) \eta(ds, dv), \quad \Phi \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E), \\ \bar{R}_\lambda h(t, x) &= \int_0^t G_{s,t}^\lambda * h(s, x) dW_s, \quad h \in \mathfrak{D}_p(E, Y).\end{aligned}$$

**Lemma 15** *Let  $\alpha \in (0, 2]$ ,  $p \geq 2$  and Assumption  $A_0$  be satisfied. Then there is a constant  $C = C(\alpha, p, d, K, \delta)$  such that the following estimates*

hold:

- (i)  $|T^\lambda u_0|_{\mathbb{L}_p(E)} \leq \rho_\lambda^{\frac{1}{p}} |u_0|_{\mathbb{L}_p(\mathbf{R}^d)}, \quad u_0 \in \mathfrak{D}_p(\mathbf{R}^d),$
- (ii)  $|R_\lambda f|_{\mathbb{L}_p(E)} \leq \rho_\lambda |f|_{\mathbb{L}_p(E)}, \quad f \in \mathfrak{D}_p(E),$
- (iii)  $|\tilde{R}_\lambda \Phi|_{\mathbb{L}_p(E)} \leq C \sum_{r=2,p} \rho_\lambda^{\frac{1}{r}} |\Phi|_{\mathbb{L}_{r,p}(E)}, \quad \Phi \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E),$
- (iv)  $|\bar{R}_\lambda h|_{\mathbb{L}_p(E)} \leq C \rho_\lambda^{\frac{1}{2}} |h|_{\mathbb{L}_p(E,Y)}, \quad h \in \mathfrak{D}_p(E,Y),$

where  $\rho_\lambda = T \wedge \frac{1}{\lambda}$ .

**Proof.** (i) By Minkowski's inequality and Remark 14,

$$\begin{aligned} |T^\lambda u_0|_{\mathbb{L}_p(E)}^p &= \mathbf{E} \int_0^T \int_{\mathbf{R}^d} |G_{0,t}^\lambda * u_0(x)|^p dx dt \leq \int_0^T e^{-\lambda p t} |u_0|_{\mathbb{L}_p(\mathbf{R}^d)}^p dt \\ &\leq \rho_\lambda |u_0|_{\mathbb{L}_p(\mathbf{R}^d)}^p. \end{aligned}$$

(ii) By Minkowski's inequality and Remark 14,

$$\begin{aligned} |R_\lambda f|_{\mathbb{L}_p(E)}^p &= \mathbf{E} \int_0^T \int_{\mathbf{R}^d} \left| \int_0^t G_{s,t}^\lambda * f(s, x) ds \right|^p dx dt \\ &\leq \int_0^T \left( \int_0^t \int_{\mathbf{R}^d} G_{s,t}^\lambda(y) |f(s, \cdot)|_{\mathbb{L}_p(\mathbf{R}^d)} dy ds \right)^p dt \\ &= \int_0^T \left( \int_0^t e^{-\lambda(t-s)} |f(s, \cdot)|_{\mathbb{L}_p(\mathbf{R}^d)} ds \right)^p dt \equiv I. \end{aligned}$$

Applying here Hölder's inequality, we get

$$\begin{aligned} I &\leq \int_0^T \left( \int_0^t e^{-\lambda(t-s)} ds \right)^{p-1} \int_0^t e^{-\lambda(t-s)} |f(s, \cdot)|_{\mathbb{L}_p(\mathbf{R}^d)}^p ds dt \\ &\leq \rho_\lambda^{p-1} \int_0^T \int_s^T e^{-\lambda(t-s)} |f(s, \cdot)|_{\mathbb{L}_p(\mathbf{R}^d)}^p dt ds \leq \rho_\lambda^p |f|_{\mathbb{L}_p(E)}^p. \end{aligned} \quad (20)$$

(iii) By Doob's and Minkowski's inequalities

$$\begin{aligned} \mathbf{E} |\bar{R}_\lambda h(t, x)|^p &= \mathbf{E} \left| \int_0^t G_{s,t}^\lambda * h(s, x) dW_s \right|^p \leq C \mathbf{E} \left( \int_0^t |G_{s,t}^\lambda * h(s, x)|_Y^2 ds \right)^{\frac{p}{2}} \\ &\leq C \mathbf{E} \left( \int_0^t [G_{s,t}^\lambda * |h(s, x)|_Y]^2 ds \right)^{\frac{p}{2}}. \end{aligned}$$

Therefore, by Minkowski's inequality and Remark 14,

$$\begin{aligned}
|\bar{R}_\lambda h|_{\mathbb{L}_p(E)}^p &= \mathbf{E} \int_0^T \int_{\mathbf{R}^d} |\bar{R}_\lambda h(t, x)|^p dx dt \\
&\leq C \mathbf{E} \int_0^T \int_{\mathbf{R}^d} \left( \int_0^t [G_{s,t}^\lambda * |h(s, x)|_Y]^2 ds \right)^{\frac{p}{2}} dx dt \\
&\leq C \int_0^T \left( \int_0^t \int_{\mathbf{R}^d} e^{-2\lambda(t-s)} G_{s,t}(y) |h(s, \cdot)|_{\mathbb{L}_p(\mathbf{R}^d, Y)}^2 dy ds \right)^{\frac{p}{2}} dt \\
&= C \int_0^T \left( \int_0^t e^{-2\lambda(t-s)} |h(s, \cdot)|_{\mathbb{L}_p(\mathbf{R}^d, Y)}^2 ds \right)^{\frac{p}{2}} dt.
\end{aligned}$$

Now, similarly as in (20) with  $p$  replaced by  $p/2$ , we get

$$|\bar{R}_\lambda h|_{\mathbb{L}_p(E)}^p \leq C \rho_\lambda^{\frac{p}{2}} |h|_{\mathbb{L}_p(E)}^p.$$

By Corollary 2 in [13],

$$|\tilde{R}_\lambda \Phi|_{\mathbb{L}_p(E)}^p \leq C(A + B),$$

where

$$A = \mathbf{E} \int_0^T \int_{\mathbf{R}^d} \left( \int_0^t \int_U [G_{s,t}^\lambda * \Phi(s, x, v)]^2 \Pi(dv) ds \right)^{\frac{p}{2}} dx dt$$

and

$$B = \mathbf{E} \int_0^T \int_{\mathbf{R}^d} \int_0^t \int_U |G_{s,t}^\lambda * \Phi(s, x, v)|^p \Pi(dv) ds dx dt.$$

By Minkowski's inequality and Remark 14,

$$\begin{aligned}
A &\leq \mathbf{E} \int_0^T \int_{\mathbf{R}^d} \left( \int_0^t \left[ G_{s,t}^\lambda * \left( \int_U \Phi^2(s, x, v) \Pi(dv) \right)^{\frac{1}{2}} \right]^2 ds \right)^{\frac{p}{2}} dx dt \\
&\leq \int_0^T \left( \int_0^t e^{-2\lambda(t-s)} |\Phi(s, \cdot, \cdot)|_{\mathbb{L}_{2,p}(\mathbf{R}^d)}^2 ds \right)^{\frac{p}{2}} dt
\end{aligned}$$

and

$$\begin{aligned}
B &\leq \mathbf{E} \int_0^T \int_{\mathbf{R}^d} \int_0^t \left[ G_{s,t}^\lambda * \left( \int_U |\Phi(s, x, v)|^p \Pi(dv) \right)^{\frac{1}{p}} \right]^p ds dx dt \\
&\leq \int_0^T \int_0^t e^{-p\lambda(t-s)} |\Phi(s, \cdot, \cdot)|_{\mathbb{L}_{p,p}(\mathbf{R}^d)}^p ds dt.
\end{aligned}$$



Now, similarly as in (20), we get

$$A \leq \rho_\lambda^{\frac{p}{2}} |\Phi|_{\mathbb{L}_{2,p}(E)}^p$$

and

$$B \leq \rho_\lambda |\Phi|_{\mathbb{L}_{p,p}(E)}^p.$$

■

**Lemma 16** *Let  $\alpha \in (0, 2]$ ,  $\beta \in \mathbf{R}$ ,  $p \geq 2$  and Assumption  $\mathbf{A}_0$  be satisfied. Then there is a constant  $C = C(\alpha, p, d, K, \delta, T)$  such that the following estimates hold:*

- (i)  $|T^\lambda u_0|_{\mathbb{H}_p^{\beta+\alpha}(E)} \leq C |u_0|_{\mathbb{B}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(\mathbf{R}^d)}, \quad u_0 \in \mathfrak{D}_p(\mathbf{R}^d),$
- (ii)  $|R_\lambda f|_{\mathbb{H}_p^{\beta+\alpha}(E)} \leq C |f|_{\mathbb{H}_p^\beta(E)}, \quad f \in \mathfrak{D}_p(E),$
- (iii)  $|\tilde{R}_\lambda \Phi|_{\mathbb{H}_p^{\beta+\alpha}(E)} \leq C \sum_{r=2,p} |\Phi|_{\mathbb{B}_{r,pp}^{\beta+\alpha-\frac{\alpha}{r}}(E)}, \quad \Phi \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E),$
- (iv)  $|\bar{R}_\lambda h|_{\mathbb{H}_p^{\beta+\alpha}(E)} \leq C |h|_{\mathbb{H}_p^{\beta+\alpha/2}(E,Y)}, \quad h \in \mathfrak{D}_p(E, Y).$

**Proof.** We have  $\mathbb{L}_p$ -estimates by Lemma 15. The estimate (ii) follows by Theorem 2.1 in [11]. The estimate (iii) is proved in [13] (we apply Corollary 1 and Proposition 2 with  $V = L_2(U, \mathcal{U}, \Pi)$ ). The estimate (iv) is proved in [13] (Proposition 2 with  $V = Y$ ).

It remains to prove (i). We follow the arguments in [13] (see [3] as well). Let

$$T_t u_0(x) = G_{0,t} * u_0(x).$$

Since

$$|T^\lambda u_0|_{\mathbb{H}_p^\beta(E)}^p = \mathbf{E} \int_0^T e^{-\lambda p t} |T_t u_0|_{H_p^\beta(\mathbf{R}^d)}^p dt \leq \mathbf{E} |T \cdot u_0|_{H_p^\beta(E)}^p,$$

it suffices to prove the estimate

$$|T \cdot u_0|_{H_p^\beta(E)} \leq C |u_0|_{\mathbb{B}_{pp}^{\beta-\frac{\alpha}{p}}(\mathbf{R}^d)} \quad (21)$$

for non-random functions  $u_0 \in \mathfrak{D}_p(\mathbf{R}^d)$ .

In order to show (21), we follow [11]. Let

$$\begin{aligned} \tilde{\varphi}_0 &= \varphi_0 + \varphi_1, \\ \tilde{\varphi}_j &= \varphi_{j-1} + \varphi_j + \varphi_{j+1}, \quad j \geq 1, \end{aligned}$$

where the functions  $\varphi_j$ ,  $j \geq 0$ , are defined in Subsection 2.2. Let

$$h_t^j(x) = G_{0,t} * \tilde{\varphi}_j(x), \quad j \geq 0.$$

According to Lemma 12 in [11] or inequality (36) and Lemma 16 in [13], there are positive constants  $C$  and  $c$  such that for all  $s < t$ ,  $j \geq 1$ ,

$$|h_t^j|_1 \leq C e^{-c2^{j\alpha}t} \sum_{k \leq d_0} (2^{j\alpha}t)^k, \quad |h_t^0|_1 \leq C. \quad (22)$$

Here and in the remaining part of the proof we use the notation  $|\cdot|_p = |\cdot|_{L_p(\mathbf{R}^d)}$ ,  $p \geq 1$ .

We set

$$u_{0,j} = u_0 * \varphi_j, \quad j \geq 0.$$

Obviously,

$$\varphi_j * T_t u_0 = T_t(u_0 * \varphi_j) = T_t u_{0,j}, \quad j \geq 0.$$

Since  $\varphi_j = \varphi_j * \tilde{\varphi}_j$ ,  $j \geq 0$ , we have

$$T_t u_{0,j} = h_t^j * u_{0,j}, \quad j \geq 0.$$

By Minkowski's inequality,

$$\begin{aligned} |T_t u_0|_{H_p^\beta(E)}^p &= \int_0^T \int_{\mathbf{R}^d} \left( \sum_{j=0}^{\infty} 2^{2\beta j} [\varphi_j * T_t u_0(x)]^2 \right)^{\frac{p}{2}} dx dt \\ &= \int_0^T \int_{\mathbf{R}^d} \left( \sum_{j=0}^{\infty} 2^{2\beta j} [h_t^j * u_{0,j}(x)]^2 \right)^{\frac{p}{2}} dx dt \\ &\leq \int_0^T \left( \sum_{j=0}^{\infty} 2^{2\beta j} |h_t^j * u_{0,j}|_p^2 \right)^{\frac{p}{2}} dt. \end{aligned} \quad (23)$$

Applying Minkowski's inequality and (22), we get

$$|h_t^j * u_{0,j}|_p \leq |h_t^j|_1 |u_{0,j}|_p \leq C e^{-c2^{j\alpha}t} |u_{0,j}|_p, \quad j \geq 0.$$

Hence, by (23)

$$|T_t u_0|_{H_p^\beta(E)}^p \leq C \int_0^T \left( \sum_{j=0}^{\infty} e^{-c2^{j\alpha}t} 2^{2\beta j} |u_{0,j}|_p^2 \right)^{\frac{p}{2}} dt. \quad (24)$$

If  $p = 2$ , we have immediately

$$\begin{aligned} |T.u_0|_{H_p^\beta(E)}^p &\leq C \int_0^T \sum_{j=0}^{\infty} e^{-c2^{\alpha j}t} 2^{2\beta j} |u_{0,j}|_2^2 dt \\ &\leq C \sum_{j=0}^{\infty} 2^{-\alpha j} 2^{2\beta j} |u_{0,j}|_2^2 = C |u_0|_{B_{22}^{\beta-\frac{\alpha}{2}}(\mathbf{R}^d)}^2. \end{aligned}$$

If  $p > 2$ , we split the sum in (24) as follows:

$$\begin{aligned} \sum_{j=0}^{\infty} e^{-c2^{\alpha j}t} 2^{2\beta j} |u_{0,j}|_p^2 &= \sum_{j \in J} e^{-c2^{\alpha j}t} 2^{2\beta j} |u_{0,j}|_p^2 \\ &+ \sum_{j \in \mathbb{N}_0 \setminus J} e^{-c2^{\alpha j}t} 2^{2\beta j} |u_{0,j}|_p^2 = A(t) + B(t), \end{aligned}$$

where  $J = \{j \in \mathbb{N}_0 : 2^{\alpha j}t \leq 1\}$ .

Fix  $\kappa \in (0, \frac{2\alpha}{p})$ . Using Hölder's inequality, we get

$$A(t) \leq \sum_{j \in J} 2^{2\beta j} 2^{\kappa j} 2^{-\kappa j} |u_{0,j}|_p^2 \leq \left( \sum_{j \in J} 2^{q\kappa j} \right)^{1/q} \left( \sum_{j \in J} 2^{p\beta j} 2^{-p\kappa j/2} |u_{0,j}|_p^p \right)^{2/p}$$

with  $q = \frac{p}{p-2}$ . Since

$$\sum_{j \in J} 2^{q\kappa j} \leq C t^{-q\kappa/\alpha},$$

we have

$$\begin{aligned} A(t) &\leq C t^{-\frac{\kappa}{\alpha}} \left( \sum_{j \in J} 2^{p\beta j} 2^{-p\kappa j/2} |u_{0,j}|_p^p \right)^{\frac{2}{p}} \\ &= C t^{-\frac{\kappa}{\alpha}} \left( \sum_j 1_{\{t \leq 2^{-\alpha j}\}} 2^{p\beta j} 2^{-p\kappa j/2} |u_{0,j}|_p^p \right)^{\frac{2}{p}}. \end{aligned}$$

So,

$$\begin{aligned} \int_0^T A(t)^{\frac{p}{2}} dt &\leq C \sum_j 2^{p\beta j} 2^{-p\kappa j/2} |u_{0,j}|_p^p \int_0^{2^{-\alpha j}} t^{-\frac{p\kappa}{2\alpha}} dt \\ &\leq C \sum_j 2^{-\alpha j} 2^{p\beta j} |u_{0,j}|_p^p. \end{aligned}$$

By Hölder's inequality,

$$B(t) \leq \left\{ \sum_{j \in \mathbb{N}_0 \setminus J} e^{-c2^{\alpha j} t} \right\}^{\frac{1}{q}} \left\{ \sum_{j \in \mathbb{N}_0 \setminus J} e^{-c2^{\alpha j} t} 2^{\beta p j} |u_{0,j}|_p^p \right\}^{\frac{2}{p}}$$

with  $q = \frac{p}{p-2}$ . Since  $e^{-c2^{\alpha j} t}$  is decreasing in  $j$ , we have

$$\sum_{j \in \mathbb{N}_0 \setminus J} e^{-c2^{\alpha j} t} \leq \int_{t \geq 2^{-\alpha r}} e^{-c2^{-\alpha} 2^{\alpha r} t} dr \leq C.$$

Therefore,

$$\begin{aligned} \int_0^T B(t)^{\frac{p}{2}} dt &\leq C \sum_j 2^{\beta p j} |u_{0,j}|_p^p \int_0^T e^{-c2^{\alpha j} t} dt \\ &\leq C \sum_j 2^{-\alpha j} 2^{\beta p j} |u_{0,j}|_p^p. \end{aligned}$$

Finally,

$$\begin{aligned} |T.u_0|_{H_p^\beta(E)}^p &\leq C \left( \int_0^T A(t)^{\frac{p}{2}} dt + \int_0^T B(t)^{\frac{p}{2}} dt \right) \\ &\leq C \sum_j 2^{-\alpha j} 2^{\beta p j} |u_{0,j}|_p^p = C |u_0|_{B_{pp}^{\beta-\frac{\alpha}{p}}(\mathbf{R}^d)}^p. \end{aligned}$$

The lemma is proved. ■

For a bounded measurable  $m(y), y \in \mathbf{R}^d$ , and  $\alpha \in (0, 2)$ , set for  $v \in \mathcal{S}(\mathbf{R}^d), x \in \mathbf{R}^d$ ,

$$\mathcal{L}v(x) = \mathcal{L}_\alpha u(x) = \int \nabla^\alpha v(x) m(y) \frac{dy}{|y|^{d+\alpha}}.$$

We will need the following continuity estimate (see [2] for a symmetric case, Theorem 2.1 in [5] for a general case using Hölder estimates, and Lemma 10 in [14] for a direct proof).

**Lemma 17** (*Lemma 10, [14]*) *Let  $|m(y)| \leq K, y \in \mathbf{R}^d, p > 1$ , and  $\alpha \in (0, 2)$ . Assume*

$$\int_{r \leq |y| \leq R} y m(y) \frac{dy}{|y|^{d+\alpha}} = 0$$

*for any  $0 < r < R$  if  $\alpha = 1$ . Then there is a constant  $C$  such that*

$$|\mathcal{L}_\alpha u|_p \leq CK |\partial^\alpha u|_p, u \in L_p(\mathbf{R}^d).$$

## 4.2 Solution for smooth input functions

**Theorem 18** *Let  $\alpha \in (0, 2]$ ,  $p \geq 2$  and Assumption  $A_0$  be satisfied. Let  $u_0 \in \mathfrak{D}_p(\mathbf{R}^d)$  be  $\mathcal{F}_0$ -measurable,  $f \in \mathfrak{D}_p(E)$ ,  $\Phi \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E)$  and  $h \in \mathfrak{D}_p(E, Y)$ .*

*Then there is a unique strong solution  $u \in \mathfrak{D}_p(E)$  of (19). Moreover,  $\mathbf{P}$ -a.s.  $u(t, x)$  is cadlag in  $t$ , smooth in  $x$  and the following assertions hold:*

(i) *for each multiindex  $\gamma \in \mathbf{N}_0^d$  and  $(t, x) \in E$   $\mathbf{P}$ -a.s.*

$$\begin{aligned} \partial_x^\gamma u(t, x) &= T_t^\lambda \partial^\gamma u_0(x) + R_\lambda \partial_x^\gamma f(t, x) + \tilde{R}_\lambda \partial_x^\gamma \Phi(t, x) \\ &\quad + \bar{R}_\lambda \partial_x^\gamma h(t, x); \end{aligned}$$

(ii) *for each multiindex  $\gamma \in \mathbf{N}_0^d$*

$$\begin{aligned} |\partial^\gamma u|_{\mathbb{L}_p(E)} &\leq C \left\{ \rho_\lambda^{1/p} |\partial^\gamma u_0|_{\mathbb{L}_p(\mathbf{R}^d)} + \rho_\lambda |\partial^\gamma f|_{\mathbb{L}_p(E)} \right. \\ &\quad \left. + \sum_{r=2,p} \rho_\lambda^{1/r} |\partial^\gamma \Phi|_{\mathbb{L}_{r,p}(E)} + \rho_\lambda^{1/2} |\partial^\gamma h|_{\mathbb{L}_p(E,Y)}^p \right\}, \end{aligned}$$

where  $\rho_\lambda = T \wedge \frac{1}{\lambda}$  and the constant  $C = C(\alpha, p, d, |\gamma|, K, \delta)$ ;

(iii) *for each  $\beta \in \mathbf{R}$*

$$\begin{aligned} |u|_{\mathbb{H}_p^{\beta+\alpha}(E)} &\leq C \left\{ |u_0|_{\mathbb{B}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(\mathbf{R}^d)} + |f|_{\mathbb{H}_p^\beta(E)} + |\Phi|_{\mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E)} \right. \\ &\quad \left. + |\Phi|_{\mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} + |h|_{\mathbb{H}_p^{\beta+\alpha/2}(E,Y)} \right\}, \end{aligned}$$

where the constant  $C = C(\alpha, \beta, p, d, T, K, \delta)$ .

**Proof.** We follow the arguments in [15], [12]. Denote by  $C_p^\infty(E)$  the set of all  $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbf{R}^d)$ -measurable random functions  $v(t, x)$  on  $E$  such that  $\mathbf{P}$ -a.s. for all  $t \in [0, T]$   $u(t, x)$  is infinitely differentiable in  $x$  and for every multiindex  $\gamma \in \mathbf{N}_0^d$

$$\sup_{(t,x) \in E} \mathbf{E} |D_x^\gamma v(t, x)|^p < \infty.$$

According to the definition of  $\mathfrak{D}_p(E)$ , we have  $\mathfrak{D}_p(E) \subset C_p^\infty(E)$ .

Let  $u_0 = 0$ . Since for every multiindex  $\gamma \in \mathbf{N}_0^d$

$$\begin{aligned} \sup_{(t,x) \in E} \mathbf{E} \left\{ |D_x^\gamma f(t, x)|^p + \sum_{r=2,p} \left( \int_U |D_x^\gamma \Phi(t, x, v)|^r \Pi(dv) \right)^{\frac{p}{r}} \right. \\ \left. + |D_x^\gamma h(t, x)|_Y^p \right\} < \infty, \end{aligned}$$

by Lemma 8 in [15] and Lemma 7 in [12] there is a unique  $u \in C_p^\infty(E)$  solving (19),  $u(t, x)$  is cadlag in  $t$  and the assertion (i) holds with  $\gamma = 0$ . Moreover (see equation (20) in the proof of Lemma 8 in [15] and the proof of Lemma 7 in [12]), for every  $\gamma \in \mathbf{N}_0^d$  and  $(t, x) \in E$  we have  $\mathbf{P}$ -a.s.

$$\begin{aligned} D_x^\gamma u(t, x) &= \int_0^t [A_0^{(\alpha)} D_x^\gamma u - \lambda D_x^\gamma u + D_x^\gamma f](s, x) ds \\ &\quad + \int_0^t \int_U D_x^\gamma \Phi(s, x, v) \eta(ds, dv) + \int_0^t D_x^\gamma h(s, x) dW_s. \end{aligned}$$

Applying Lemma 8 in [15] and Lemma 7 in [12] again, we get the assertion (i) for arbitrary  $\gamma \in \mathbf{N}_0^d$ . The estimates (ii) and (iii) follow by Lemmas 15 and 16. The assertion (iii) and embedding theorem imply that  $u \in \mathfrak{D}_p(E)$ . Using Lemma 3.2 in [11], we get that there is a constant  $C$  such that for every  $v \in \mathbb{H}_p^{\beta+\alpha}(E)$

$$|A_0^{(\alpha)} v|_{\mathbb{H}_p^\beta(E)} \leq C |v|_{\mathbb{H}_p^{\beta+\alpha}(E)}.$$

Hence,  $u \in \mathfrak{D}_p(E)$  is a unique strong solution of (19).

The case  $u_0 \neq 0$  is considered as above repeating the proof of Lemma 8 in [15] with obvious changes. The theorem is proved.  $\blacksquare$

## 5 Model problem. Partial case II

In this section, we consider the following partial case of equation (1):

$$\begin{aligned} du(t, x) &= (A^{(\alpha)} u - \lambda u + f)(t, x) dt + \int_U \Phi(t, x, v) \eta(dt, dv) \\ &\quad + \int_{\mathbf{R}_0^d} g(t, x, y) q^{(\alpha)}(dt, dy) 1_{\alpha \in (0, 2)} + h(t, x) dW_t, \\ u(0, x) &= u_0(x). \end{aligned} \tag{25}$$

We prove Theorem 5 which is a partial case of the following statement.

**Theorem 19** *Let  $\alpha \in (0, 2]$ ,  $\beta \in \mathbf{R}$ ,  $p \geq 2$  and Assumptions A(i)-(ii) be satisfied with  $\sigma^i = 0, i = 1, \dots, d$ . Assume  $\mathbf{P}$ -a.s. for all  $t \in [0, T]$  and  $y \in \mathbf{R}_0^d$*

$$\begin{aligned} m^{(\alpha)}(t, y) &\geq m_0^{(\alpha)}(t, y) \quad \text{if } \alpha \in (0, 2), \\ (B^{ij}(t)) y_i y_j &\geq \delta |y|^2 \quad \text{if } \alpha = 2, \end{aligned}$$

where the functions  $m_0^{(\alpha)}$  satisfy Assumption  $A_0$ . Let  $u_0 \in \mathbb{B}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(\mathbf{R}^d)$  be  $\mathcal{F}_0$ -measurable,  $f \in \mathbb{H}_p^\beta(E)$ ,  $\Phi \in \mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E) \cap \mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E)$ ,  $g \in \mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E) \cap \mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E)$  and  $h \in \mathbb{H}_p^{\beta+\alpha/2}(E, Y)$ .

Then there is a unique strong solution  $u \in \mathbb{H}_p^{\beta+\alpha}(E)$  of (25). Moreover, there is a constant  $C = C(\alpha, \beta, p, d, T, K, \delta)$  such that

$$\begin{aligned} |u|_{\mathbb{H}_p^{\beta+\alpha}(E)} \leq & C \left( |u_0|_{\mathbb{B}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} + |f|_{\mathbb{H}_p^\beta(E)} + |\Phi|_{\mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E)} + |\Phi|_{\mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} \right. \\ & \left. + (|g|_{\mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E)} + |g|_{\mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)}) + |h|_{\mathbb{H}_p^{\beta+1}(E,Y)} \right). \end{aligned} \quad (26)$$

First, we consider (25) for smooth in  $x$  input functions  $u_0, f, \Phi, g$  and  $h$ .

**Lemma 20** Let  $\alpha \in (0, 2]$ ,  $\beta \in \mathbf{R}$ ,  $p \geq 2$  and Assumption A be satisfied with  $\sigma^i = 0$ ,  $i = 1, \dots, d$ , and  $l^{(\alpha)} = 0$  in A(iii). Let  $u_0 \in \mathfrak{D}_p(\mathbf{R}^d)$  be  $\mathcal{F}_0$ -measurable,  $f \in \mathfrak{D}_p(E)$ ,  $\Phi \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E)$ ,  $g \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E)$  and  $h \in \mathfrak{D}_p(E, Y)$ .

Then there is a unique strong solution  $u \in \mathfrak{D}_p(E)$  of (25). Moreover,  $\mathbf{P}$ -a.s.  $u(t, x)$  is cadlag in  $t$ , smooth in  $x$  and the following assertions hold:

(i) for every multiindex  $\gamma \in \mathbf{N}_0^d$

$$\begin{aligned} |D^\gamma u|_{\mathbb{L}_p(E)} \leq & C \left\{ \rho_\lambda^{1/p} |D^\gamma u_0|_{\mathbb{L}_p(\mathbf{R}^d)} + \rho_\lambda |D^\gamma f|_{\mathbb{L}_p(E)} \right. \\ & + \sum_{r=2,p} \rho_\lambda^{1/r} (|D^\gamma \Phi|_{\mathbb{L}_{r,p}(E)} + |D^\gamma g|_{\mathbb{L}_{r,p}(E)}) \\ & \left. + \rho_\lambda^{1/2} |D^\gamma h|_{\mathbb{L}_p(E,Y)} \right\}. \end{aligned} \quad (27)$$

where  $\rho_\lambda = T \wedge \frac{1}{\lambda}$  and the constant  $C = C(\alpha, p, d, |\gamma|, K, \delta)$ ;

(ii) the estimate (26) holds for every  $\beta \in \mathbf{R}$ .

**Proof.** *Existence.*  $1^0$ . First, we consider the equation (25) with  $A^{(\alpha)}$  replaced by  $A_0^{(\alpha)}$  (equivalently,  $m^{(\alpha)}$  in the definition of  $A^{(\alpha)}$  is replaced by  $m_0^{(\alpha)}$ ).

By Lemma 14.50 and Theorem 14.56 in [7], there is a  $\mathbf{R}^d$ -valued  $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbf{R}_0)$ -measurable random function  $c^{(\alpha)}(t, z)$  on  $[0, T] \times \mathbf{R}_0$  satisfying (14) and a Poisson point measure  $\tilde{p}(dt, dz)$  on  $([0, \infty) \times \mathbf{R}_0, \mathcal{B}([0, \infty)) \otimes \mathcal{B}(\mathbf{R}_0))$ , possibly on an extended probability space, such that

$$p^{(\alpha)}(dt, dy) = \int_{\mathbf{R}_0} 1_{dy}(c^{(\alpha)}(t, z)) \tilde{p}(dt, dz)$$

and

$$q^{(\alpha)}(dt, dy) = \int_{\mathbf{R}_0} 1_{dy}(c^{(\alpha)}(t, z)) \tilde{q}(dt, dz),$$

where  $\tilde{q}(dt, dz) = \tilde{p}(dt, dz) - \frac{dzdt}{z^2}$ . Hence, for every  $g \in \bar{\mathfrak{D}}_{2,p}(E) \cap \bar{\mathfrak{D}}_{p,p}(E)$ , we have

$$\int_0^t \int_{\mathbf{R}_0^d} g(s, x, y) q^{(\alpha)}(ds, dy) = \int_0^t \int_{\mathbf{R}_0} \tilde{g}(s, x, z) \tilde{q}(ds, dz),$$

where  $\tilde{g}(s, x, z) = g(s, x, c^{(\alpha)}(s, z))$ . Since the point measures  $\tilde{p}$  and  $\eta$  have no common jumps, the problem (25) reduces to the case of a single point measure on  $[0, \infty) \times W$ , where  $W$  is the sum of  $U$  and  $\mathbf{R}_0$ . Therefore, Theorem 18 applies and all the assertions of the Lemma follow in the case  $A^{(\alpha)} = A_0^{(\alpha)}$ .

2<sup>0</sup>. Let  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}})$  be a complete probability space with a filtration of  $\sigma$ -algebras  $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)$  satisfying the usual conditions. Let  $\bar{p}(dt, dz)$  be an  $\bar{\mathbb{F}}$ -adapted Poisson measure on  $([0, \infty) \times \mathbf{R}_0, \mathcal{B}([0, \infty)) \otimes \mathcal{B}(\mathbf{R}_0))$  with the compensator  $dt dz / z^2$  and  $\bar{W}_t$  be an independent standard  $\bar{\mathbb{F}}$ -adapted Wiener process in  $\mathbf{R}^d$ .

We introduce the product of probability spaces

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}) = (\Omega \times \bar{\Omega}, \mathcal{F} \otimes \bar{\mathcal{F}}, \mathbf{P} \times \bar{\mathbf{P}}).$$

Let  $\tilde{\mathcal{F}}'$  be the completion of  $\tilde{\mathcal{F}}$ . Let  $\tilde{\mathbb{F}}' = (\tilde{\mathcal{F}}'_t)$ ,  $\tilde{\mathbb{F}}'' = (\tilde{\mathcal{F}}''_t)$  and  $\tilde{\mathbb{F}}''' = (\tilde{\mathcal{F}}'''_t)$  be the usual augmentations of  $(\mathcal{F}_t \otimes \bar{\mathcal{F}}_t)$ ,  $(\mathcal{F} \otimes \bar{\mathcal{F}}_t)$  and  $(\mathcal{F}_t \otimes \bar{\mathcal{F}})$ , respectively (see [4]).

Obviously,

$$\bar{q}(dt, dz) = \bar{p}(dt, dz) - \frac{dzdt}{z^2}$$

is an  $(\tilde{\mathbb{F}}', \tilde{\mathbf{P}})$ - and  $(\tilde{\mathbb{F}}'', \tilde{\mathbf{P}})$ -martingale measure. Also,  $q^{(\alpha)}(dt, dy)$  and  $\eta(dt, dv)$  are  $(\tilde{\mathbb{F}}', \tilde{\mathbf{P}})$ - and  $(\tilde{\mathbb{F}}''', \tilde{\mathbf{P}})$ -martingale measures.

By Lemma 14.50 in [7], there is a  $\mathcal{R}(\mathbb{F}) \otimes \mathcal{B}(\mathbf{R}_0)$ -measurable  $\mathbf{R}^d$ -valued function  $c_0^{(\alpha)}(t, z)$  such that

$$[m^{(\alpha)}(t, y) - m_0^{(\alpha)}(t, y)] \frac{dy}{|y|^{d+\alpha}} = \int_{\mathbf{R}_0} 1_{dy}(c_0^{(\alpha)}(t, z)) \frac{dz}{z^2}$$

with  $\alpha \in (0, 2)$ .



Let  $\sigma_\delta(t)$  be a symmetric square root of the matrix  $B(t) - \delta I$ . We introduce the processes  $Y_t^{(\alpha)}$ ,  $t \in [0, T]$ , defined by

$$\begin{aligned} Y_t^{(\alpha)} &= \int_0^t \int_{\mathbf{R}_0} \chi^{(\alpha)}(c_0^{(\alpha)}(s, z)) c_0^{(\alpha)}(s, z) \bar{q}(ds, dz) \\ &\quad + \int_0^t \int_{\mathbf{R}_0} [1 - \chi^{(\alpha)}(c_0^{(\alpha)}(s, z))] c_0^{(\alpha)}(s, z) \bar{p}(ds, dz) \end{aligned}$$

for  $\alpha \in (0, 2)$  and

$$Y_t^{(2)} = \int_0^t \sigma_\delta(s) d\bar{W}_s.$$

Let us consider the problem

$$\begin{aligned} dw(t, x) &= [A_0^{(\alpha)} w(t, x) - \lambda w(t, x) + f(t, x - Y_t^{(\alpha)})] dt \\ &\quad + \int_U \Phi(t, x - Y_{t-}^{(\alpha)}, v) \eta(dt, dv) \\ &\quad + \int_{\mathbf{R}_0^d} g(t, x - Y_{t-}^{(\alpha)}, y) q^{(\alpha)}(dt, dy) 1_{\alpha \in (0, 2)} \\ &\quad + h(t, x - Y_t^{(\alpha)}) dW_t, \\ w(0, x) &= u_0(x). \end{aligned} \tag{28}$$

Obviously,  $f(t, x - Y_t^{(\alpha)}) \in \mathfrak{D}_p(E)$ ,  $\Phi(t, x - Y_{t-}^{(\alpha)}, v) \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E)$ ,  $g(t, x - Y_{t-}^{(\alpha)}, y) \in \tilde{\mathfrak{D}}_{2,p}(E) \cap \tilde{\mathfrak{D}}_{p,p}(E)$  and  $h(t, x - Y_t^{(\alpha)}) \in \mathfrak{D}_p(E, Y)$ , where the classes  $\mathfrak{D}_p(E)$ ,  $\mathfrak{D}_{r,p}(E)$ ,  $\tilde{\mathfrak{D}}_{r,p}(E)$  and  $\mathfrak{D}_p(E, Y)$  are defined on the extended probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$  with the filtration  $\tilde{\mathbb{F}}'$ . According to the first part of the proof, there is a unique strong solution  $w \in \mathfrak{D}_p(E)$  of (28). Moreover,  $w(t, x)$  is cadlag in  $t$ , smooth in  $x$  and possesses the properties (i) and (ii) with all the norms defined on the extended probability space. Since the norms entering the estimates (i) and (ii) are invariant with respect to random shifts of the space variable  $x \in \mathbf{R}^d$ , we conclude that the norms  $|\partial^\gamma w|_{\mathbb{L}_p(E)}$  and  $|w|_{\mathbb{H}_p^{\beta+\alpha}(E)}$  defined on the extended probability space do not exceed the right-hand sides of the estimates (i) and (ii) defined on the original probability space.

Applying the Ito-Wentzel formula (see Proposition 1 of [10]), we have

$$\begin{aligned}
w(t, x + Y_t^{(\alpha)}) &= u_0(x) + \int_0^t \left[ A_0^{(\alpha)} w(s, x + Y_s^{(\alpha)}) - \lambda w(s, x + Y_s^{(\alpha)}) \right. \\
&\quad \left. + f(s, x) + \frac{1}{2} (B(s) - \delta I)^{ij} w_{x_i x_j}(s, x + Y_s^{(\alpha)}) 1_{\alpha=2} \right] ds \\
&\quad + \int_0^t \nabla w(s-, x + Y_{s-}^{(\alpha)}) dY_s^{(\alpha)} \\
&\quad + \sum_{s \leq t} \left[ w(s-, x + Y_s^{(\alpha)}) - w(s-, x + Y_{s-}^{(\alpha)}) \right. \\
&\quad \left. - (\nabla w(s-, x + Y_{s-}^{(\alpha)}), Y_s^{(\alpha)} - Y_{s-}^{(\alpha)}) \right] 1_{\alpha \in (0,2)} \\
&\quad + \int_0^t \int_U \Phi(s, x, v) \eta(ds, dv) \\
&\quad + \int_0^t \int_{\mathbf{R}_0^d} g(s, x, y) q^{(\alpha)}(ds, dy) 1_{\alpha \in (0,2)} + \int_0^t h(s, x) dW_s.
\end{aligned}$$

Thus

$$\begin{aligned}
&w(t, x + Y_t^{(\alpha)}) \tag{29} \\
&= u_0(x) + \int_0^t \left[ A^{(\alpha)} w(s, x + Y_s^{(\alpha)}) - \lambda w(s, x + Y_s^{(\alpha)}) + f(s, x) \right] ds \\
&\quad + \int_0^t \int_{\mathbf{R}_0} \left[ w(s-, x + Y_{s-}^{(\alpha)} + c_0^{(\alpha)}(s, z)) - w(s-, x + Y_{s-}^{(\alpha)}) \right] \bar{q}(ds, dz) 1_{\alpha \in (0,2)} \\
&\quad + \int_0^t \nabla w(s, x + Y_s^{(\alpha)}) \sigma_\delta(s) d\bar{W}_s 1_{\alpha=2} + \int_0^t \int_U \Phi(s, x, v) \eta(ds, dv) \\
&\quad + \int_0^t \int_{\mathbf{R}_0^d} g(s, x, y) q^{(\alpha)}(ds, dy) 1_{\alpha \in (0,2)} + \int_0^t h(s, x) dW_s.
\end{aligned}$$

Let

$$\tilde{w}(t, x) = w(t, x + Y_t^{(\alpha)}), \quad u(t, x) = \bar{\mathbf{E}} \tilde{w}(t, x),$$

where for a measurable integrable function  $F$  on  $\tilde{\Omega} = \Omega \times \bar{\Omega}$  we denote

$$\bar{\mathbf{E}} F = \int F(\omega, \bar{\omega}) \bar{\mathbf{P}}(d\bar{\omega}).$$

Obviously,  $u \in \mathfrak{D}_p(E)$ , and by Hölders inequality,

$$|\partial^\gamma u|_{\mathbb{L}_p(E)} \leq |\partial^\gamma \tilde{w}|_{\mathbb{L}_p(E)}, \quad |u|_{\mathbb{H}_p^{\beta+\alpha}(E)} \leq |\tilde{w}|_{\mathbb{H}_p^{\beta+\alpha}(E)},$$

where the norms  $|\partial^\gamma \tilde{w}|_{\mathbb{L}_p(E)}$  and  $|\partial^\gamma \tilde{w}|_{\mathbb{H}_p^{\beta+\alpha}(E)}$  defined on the extended probability space coincide with the norms  $|\partial^\gamma w|_{\mathbb{L}_p(E)}$  and  $|w|_{\mathbb{H}_p^{\beta+\alpha}(E)}$  and do not exceed the right-hand sides of the estimates (i) and (ii). Therefore, the function  $u$  satisfies the estimates (i) and (ii). Moreover,  $u$  is cadlag in  $t$ , and smooth in  $x$ . Taking expectation  $\bar{\mathbf{E}}$  of both sides of (29), we see that  $u$  satisfies (25).

*Uniqueness.* Let  $u_i \in \mathfrak{D}_p(E)$ ,  $i = 1, 2$ , be two strong solutions of (25). Then  $u = u_1 - u_2$  is a strong solution to the problem

$$\begin{aligned} du(t, x) &= (A^{(\alpha)}u - \lambda u)(t, x)dt \quad \text{in } E, \\ u(0, x) &= 0 \quad \text{in } \mathbf{R}^d. \end{aligned} \quad (30)$$

Considering (30) separately for every  $\omega \in \Omega$ , without loss of generality we can assume that the coefficients  $m^{(\alpha)}, B, b$  of the operator  $A^{(\alpha)}$  and the function  $u$  are non-random.

We fix arbitrary  $(t_0, x) \in E$  and introduce the processes  $Z_t^{(\alpha)}$ ,  $t \in [0, t_0]$ ,  $\alpha \in (0, 2]$ , defined on some probability space by

$$\begin{aligned} Z_t^{(\alpha)} &= \int_0^t \int_{\mathbf{R}_0^d} \chi^{(\alpha)}(y) y q_\alpha(ds, dy) + \int_0^t \int_{\mathbf{R}_0^d} [1 - \chi^{(\alpha)}(y)] y p_\alpha(ds, dy) \\ &\quad + \int_0^t b(s) ds \mathbf{1}_{\alpha=1} \end{aligned}$$

for  $\alpha \in (0, 2)$  and

$$Z_t^{(2)} = \int_0^t \hat{\sigma}(s) d\widehat{W}_s.$$

Here,  $p_\alpha(dt, dy)$  is a Poisson point measure on  $([0, t_0] \times \mathbf{R}_0^d, \mathcal{B}([0, t_0]) \otimes \mathcal{B}(\mathbf{R}_0^d))$  with the compensator  $m^{(\alpha)}(t_0 - t, y) dy dt / |y|^{d+\alpha}$ ,

$$q_\alpha(dt, dy) = p_\alpha(dt, dy) - m^{(\alpha)}(t_0 - t, y) \frac{dy dt}{|y|^{d+\alpha}}$$

is a martingale measure,  $\hat{\sigma}(t)$  is a symmetric square root of the matrix  $B(t)$  and  $\widehat{W}_t$  is a standard Wiener process in  $\mathbf{R}^d$ . By Ito's formula,

$$\begin{aligned} -u(t_0, x) &= e^{-\lambda t_0} u(0, x + Z_{t_0}^{(\alpha)}) - u(t_0, x) \\ &= \int_0^{t_0} e^{-\lambda t} \left( -\frac{\partial u}{\partial t} + A^{(\alpha)}u - \lambda u \right) (t_0 - t, x + Z_t^{(\alpha)}) dt = 0. \end{aligned}$$

Since  $(t_0, x) \in E$  was arbitrary,  $u = 0$  on  $E$ .

The lemma is proved. ■

**Proof of Theorem 19.** *Existence.* According to Lemma 11, there is a sequence of input functions  $(u_{0n}, f_n, \Phi_n, g_n, h_n)$ ,  $n = 1, 2, \dots$ , such that  $u_{0n} \in \mathfrak{D}_p(\mathbf{R}^d)$ ,  $f_n \in \mathfrak{D}_p(E)$ ,  $\Phi_n \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E)$ ,  $g_n \in \overline{\mathfrak{D}}_{2,p}(E) \cap \overline{\mathfrak{D}}_{p,p}(E)$ ,  $h_n \in \mathfrak{D}_p(E, Y)$  and

$$\begin{aligned} & |u_0 - u_{0n}|_{\mathbb{B}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(\mathbf{R}^d)} + |f - f_n|_{\mathbb{H}_p^\beta(E)} + |\Phi - \Phi_n|_{\mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E)} + |\Phi - \Phi_n|_{\mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} \\ & + |g - g_n|_{\mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E)} + |g - g_n|_{\mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} + |h - h_n|_{\mathbb{H}_p^{\beta+1}(E)} \rightarrow 0 \end{aligned} \quad (31)$$

as  $n \rightarrow \infty$ . By Lemma 20, for every  $n$  there is a strong solution  $u_n \in \mathfrak{D}_p(E)$  of (25) with the input functions  $u_{0n}, f_n, \Phi_n, g_n, h_n$ . Since (25) is a linear equation, using the estimate (ii) of Lemma 20 we derive that  $(u_n)$  is a Cauchy sequence in  $\mathbb{H}_p^{\beta+\alpha}(E)$ . Hence, there is a function  $u \in \mathbb{H}_p^{\beta+\alpha}(E)$  such that  $|u_n - u|_{\mathbb{H}_p^{\beta+\alpha}(E)} \rightarrow 0$  as  $n \rightarrow \infty$ .

Passing to the limit in (26) with  $u, u_0, f, \Phi, g, h$  replaced by  $u_n, u_{0n}, f_n, \Phi_n, g_n, h_n$  and using (21), we get the estimate (26).

Passing to the limit in the equality (see Definition 3)

$$\begin{aligned} \langle J^\beta u_n(t, \cdot), \varphi \rangle &= \langle J^\beta u_0, \varphi \rangle + \int_0^t \left[ \langle A^{(\alpha)} J^\beta u_n(s, \cdot), \varphi - \lambda \varphi \rangle + \langle J^\beta f(s, \cdot), \varphi \rangle \right] ds \\ &+ \int_0^t \int_U \langle J^\beta \Phi_n(s, \cdot, v), \varphi \rangle \eta(ds, dv) \\ &+ \int_0^t \int_{\mathbf{R}_0^d} \langle J^\beta g_n(s, \cdot, y), \varphi \rangle q^{(\alpha)}(ds, dy) 1_{\alpha \in (0,2)} \\ &+ \int_0^t \langle J^\beta h_n(s, \cdot), \varphi \rangle dW_s, \quad \varphi \in \mathcal{S}(\mathbf{R}^d), \end{aligned}$$

as  $n \rightarrow \infty$  and using Lemma 17, we get that the function  $u$  is a strong solution of (25).

*Uniqueness.* Let  $u \in \mathbb{H}_p^{\beta+\alpha}(E)$  be a strong solution of (25) with zero input functions  $u_0, f, \Phi, g$  and  $h$ . Hence, for every  $\varphi \in \mathcal{S}(\mathbf{R}^d)$  and  $t \in [0, T]$  **P**-a.s.

$$\langle J^\beta u(t, \cdot), \varphi \rangle = \int_0^t \langle A^{(\alpha)} J^\beta u(s, \cdot), \varphi - \lambda \varphi \rangle ds \quad (32)$$

Let  $\zeta_\varepsilon = \zeta_\varepsilon(x)$ ,  $x \in \mathbf{R}^d$ ,  $\varepsilon \in (0, 1)$ , be the functions introduced in Section 3. Inserting  $\varphi(\cdot) = \zeta_\varepsilon(x - \cdot)$  into (32), we get that the function

$$v_\varepsilon(t, x) = J^\beta u(t, \cdot) * \zeta_\varepsilon(x)$$

belongs to  $\mathfrak{D}_p(E)$  and

$$v_\varepsilon(t, x) = \int_0^t (A^{(\alpha)} - \lambda) v_\varepsilon(s, x) ds.$$

By Lemma 20,  $v_\varepsilon = 0$   $\mathbf{P}$ -a.s. in  $E$  for all  $\varepsilon \in (0, 1)$ . Hence, for every  $\varphi \in \mathcal{S}(\mathbf{R}^d)$  and  $t \in [0, T]$   $\mathbf{P}$ -a.s.

$$0 = \langle v_\varepsilon(t, \cdot), \varphi \rangle = \langle J^\beta u(t, \cdot) * \zeta_\varepsilon, \varphi \rangle \rightarrow \langle J^\beta u(t, \cdot), \varphi \rangle$$

as  $\varepsilon \rightarrow 0$ .

The theorem is proved.  $\blacksquare$

## 6 General model

Finally let us consider the equation (1). First we solve it for the smooth input functions. For  $g \in \bar{\mathfrak{D}}_{2,p}(E) \cap \bar{\mathfrak{D}}_{p,p}(E)$  let

$$\Lambda g(t, x, y) = g(t, x - y, y), (t, x) \in E, y \in \mathbf{R}_0^d.$$

We define for  $\varepsilon > 0$

$$I_\varepsilon g(t, x) = 1_{\alpha \in (0, 2)} \int_{|y| > \varepsilon} [\Lambda g(t, x, y) - g(t, x, y)] l^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}}, (t, x) \in E.$$

If  $g, \Lambda g \in \bar{\mathfrak{D}}_{2,p}(E) \cap \bar{\mathfrak{D}}_{p,p}(E)$ , then for each  $\varepsilon > 0$  we have  $I_\varepsilon g \in \bar{\mathfrak{D}}_{2,p}(E) \cap \bar{\mathfrak{D}}_{p,p}(E)$ .

**Proposition 21** *Let  $p \geq 2$  and Assumption A hold. Let*

$$\begin{aligned} f &\in \mathfrak{D}_p(E), \Phi \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E), \\ \Lambda g, g &\in \bar{\mathfrak{D}}_{2,p}(E) \cap \bar{\mathfrak{D}}_{p,p}(E), u_0 \in \mathfrak{D}_p(\mathbf{R}^d) \end{aligned}$$

( $u_0$  is  $\mathcal{F}_0$ -measurable). Assume that there is  $Ig \in \mathfrak{D}_p(E)$  such that for every  $\kappa \in \mathbf{R}$  and multiindex  $\gamma \in \mathbf{N}_0^d$

$$|I_\varepsilon g - Ig|_{\mathbb{H}_p^\kappa(E)} + \mathbf{E} \left[ \sup_{(s,x) \in E} |D_x^\gamma I_\varepsilon g(s, x) - D_x^\gamma Ig(s, x)|^p \right] \rightarrow 0$$

as  $\varepsilon \rightarrow 0$  (we denote

$$Ig(t, x) = 1_{\alpha \in (0, 2)} \int [g(t, x - y, y) - g(t, x, y)] l^{(\alpha)}(t, y) \frac{dy}{|y|^{d+\alpha}}, (t, x) \in E.)$$

Then there is a unique  $u \in \mathfrak{D}_p(E)$  solving (1). Moreover,  $\mathbf{P}$ -a.s.  $u(t, x)$  is cadlag in  $t$  and smooth in  $x$ , and there is a constant  $C$  independent of  $u_0, f, g, \Phi$  such that

$$\begin{aligned} |u|_{\mathbb{H}_p^{\beta+\alpha}(E)} \leq & C[|u_0|_{\mathbb{B}_p^{\beta+\alpha-\frac{\alpha}{p}}(\mathbf{R}^d)} + |f - Ig|_{\mathbb{H}_p^\beta(E)} + |\Phi|_{\mathbb{H}_p^{\beta+\frac{\alpha}{2}}(E)} + |\Phi|_{\mathbb{B}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} \\ & + |\Lambda g|_{\mathbb{H}_p^{\beta+\frac{\alpha}{2}}(E)} + |\Lambda g|_{\mathbb{B}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)}]. \end{aligned} \quad (33)$$

**Proof.** Let

$$\begin{aligned} Y_t^{(\alpha)} = & 1_{\alpha \in (0,2)} \left[ \int_0^t \int \chi^{(\alpha)}(y) y q^{(\alpha)}(ds, dy) + \int_0^t \int (1 - \chi^{(\alpha)}(y)) y p(ds, dy) \right] \\ & + 1_{\alpha=2} \int_0^t \sigma(s) dW_s. \end{aligned}$$

Consider the problem

$$\begin{aligned} dw(t, x) = & \{ \tilde{A}^{(\alpha)} w(t, x) - \lambda w(t, x) + f(t, x - Y_t^{(\alpha)}) - Ig(t, x - Y_t^{(\alpha)}) \} \quad (34) \\ & + \{ h(t, x - Y_t^{(\alpha)}) dW_t - 1_{\alpha=2} \partial_i h(t, x - Y_t^{(\alpha)}) \sigma^i(t) dt \} \\ & + \int g(t, x - y - Y_{t-}^{(\alpha)}, y) q^{(\alpha)}(dt, dy) + \int \Phi(t, x - Y_{t-}^{(\alpha)}, v) \eta(dt, dv), \\ w(0, x) = & u_0(x), \end{aligned}$$

where  $\tilde{A}^{(\alpha)} u$  is defined as  $A^{(\alpha)} u$  in (2) with  $m^{(\alpha)}$  replaced by  $m^{(\alpha)} - l^{(\alpha)}$  and  $B^{ij}(t)$  replaced by  $B^{ij}(t) - \frac{1}{2} \sigma^i(t) \cdot \sigma^j(t)$ . Obviously,

$$\begin{aligned} \Phi(t, x - Y_t^{(\alpha)}, v) & \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E), f(t, x - Y_t^{(\alpha)}), \\ Ig(t, x - Y_t^{(\alpha)}), \partial_i h^i(t, x - Y_t^{(\alpha)}) \sigma^i(t) 1_{\alpha=2} & \in \mathfrak{D}_p(E), \\ g(t, x - y - Y_{t-}^{(\alpha)}, y) & \in \bar{\mathfrak{D}}_{2,p}(E) \cap \bar{\mathfrak{D}}_{p,p}(E), u_0 \in \mathfrak{D}_p(\mathbf{R}^d), \end{aligned}$$

and by Lemma 20 there is a unique solution  $w \in \mathfrak{D}_p(E)$  of (34). Moreover,  $\mathbf{P}$ -a.s.  $w(t, x)$  is cadlag in  $t$ , smooth in  $x$  and the estimates (26), (27) hold. For  $\varepsilon \in (0, 1)$  set

$$\begin{aligned} Y_t^{(\alpha), \varepsilon} = & 1_{\alpha \in (0,2)} \left[ \int_0^t \int_{|y| > \varepsilon} \chi^{(\alpha)}(y) y q^{(\alpha)}(ds, dy) + \int_0^t \int (1 - \chi^{(\alpha)}(y)) y p(ds, dy) \right] \\ & + 1_{\alpha=2} \int_0^t \sigma(s) dW_s, \bar{Y}_t^{(\alpha), \varepsilon} = 1_{\alpha \in (0,2)} \int_0^t \int_{|y| \leq \varepsilon} \chi^{(\alpha)}(y) y q^{(\alpha)}(ds, dy), \end{aligned}$$

$0 \leq t \leq T$ . Applying Ito-Wentzel formula (see Proposition 1 of [10]) we have

$$\begin{aligned}
& w(t, x + Y_t^{(\alpha), \varepsilon}) \\
&= u_0(x) + \int_0^t \nabla w(s-, x + Y_{s-}^{(\alpha), \varepsilon}) dY_s^{(\alpha), \varepsilon} + \int \Phi(t, x - \bar{Y}_{s-}^{(\alpha), \varepsilon}, v) \eta(ds, dv), \\
&+ \sum_{s \leq t} [w(s-, x + Y_s^{(\alpha), \varepsilon}) - w(s-, x + Y_{s-}^{(\alpha), \varepsilon}) - \nabla w(s-, x + Y_{s-}^{(\alpha), \varepsilon}) \Delta Y_s^{(\alpha), \varepsilon}] \\
&+ \int_0^t h(s, x - \bar{Y}_{s-}^{(\alpha), \varepsilon}) dW_s + \int_0^t \int g(s, x - \bar{Y}_{s-}^{(\alpha), \varepsilon} - y, y) q^{(\alpha)}(ds, dy) \\
&+ \int_0^t \left( \tilde{A}^{(\alpha)} w(s, x + Y_s^{(\alpha), \varepsilon}) - \lambda w(s, x + Y_s^{(\alpha), \varepsilon}) + f(s, x - \bar{Y}_{s-}^{(\alpha), \varepsilon}) - I g(s, x - \bar{Y}_s^{(\alpha)}) \right) ds \\
&+ \sum_{s \leq t} [\Delta w(s, x + Y_s^{(\alpha), \varepsilon}) - \Delta w(s, x + Y_{s-}^{(\alpha), \varepsilon})] \\
&+ 1_{\alpha=2} \frac{1}{2} \int_0^t \sigma^i(s) \cdot \sigma^j(s) \partial_{ij}^2 w(s, x + Y_s^{(\alpha), \varepsilon}) ds, 0 \leq t \leq T.
\end{aligned}$$

Since

$$\begin{aligned}
& \sum_{s \leq t} [\Delta w(s, x + Y_s^{(\alpha), \varepsilon}) - \Delta w(s, x + Y_{s-}^{(\alpha), \varepsilon})] \\
&= \int_0^t \int_{|y| > \varepsilon} [g(s, x - \bar{Y}_{s-}^{(\alpha), \varepsilon}, y) - g(s, x - \bar{Y}_{s-}^{(\alpha), \varepsilon} - y, y)] p^{(\alpha)}(ds, dy) \\
&= \int_0^t \int_{|y| > \varepsilon} [g(s, x - \bar{Y}_{s-}^{(\alpha), \varepsilon}, y) - g(s, x - \bar{Y}_{s-}^{(\alpha), \varepsilon} - y, y)] q^{(\alpha)}(ds, dy) + \int_0^t I_\varepsilon g(s, x - \bar{Y}_{s-}^{(\alpha), \varepsilon}) ds,
\end{aligned}$$

it follows (by passing to the limit as  $\varepsilon \rightarrow 0$ ) that  $u(t, x) = w(t, x + Y_t^{(\alpha)})$  satisfies (1). By our assumptions and Lemma 20 (the estimate (27)),

$$|\partial^\gamma u|_{\mathbb{L}_p(E)} < \infty, \gamma \in \mathbf{N}_0^d$$

and (33) holds. Therefore  $u$  is a solution of (1). The uniqueness follows from the fact that we can go backwards. Repeating the arguments as above we find that if  $u \in \mathfrak{D}_p(E)$  solves (1) then  $w(t, x) = u(t, x - Y_t^{(\alpha)})$  is the solution of the class  $\mathfrak{D}_p(E)$  to (34) for which the uniqueness holds. ■

**Corollary 22** *There is at most one solution  $u \in \mathbb{H}_p^{\beta+\alpha}(E)$  of (1).*

**Proof.** Let  $u \in \mathbb{H}_p^{\beta+\alpha}(E)$  be a solution to (25) with zero input functions. Let  $\zeta \in C_0^\infty(\mathbf{R}^d)$ ,  $\varepsilon > 0$ ,  $\zeta_\varepsilon(x) = \varepsilon^{-d} \zeta(x/\varepsilon)$  and applying (8) with  $\zeta_\varepsilon(x - \cdot) \in C_0^\infty$

we see that

$$u_\varepsilon(t, x) = \int J^\beta u(t, y) \zeta_\varepsilon(x - y) dy$$

belongs to  $\mathfrak{D}_p(E)$  and solves (1)). Therefore, by Proposition 21  $u_\varepsilon(t, x) = 0$  for all  $\varepsilon > 0$ . The statement follows. ■

### 6.1 Proof of Theorem 8

By Lemmas 10 and 11 there are sequences

$$\begin{aligned} f_n &\in \mathfrak{D}_p(E), \Phi_n \in \mathfrak{D}_{2,p}(E) \cap \mathfrak{D}_{p,p}(E), \\ g_n &\in \bar{\mathfrak{D}}_{2,p}(E) \cap \bar{\mathfrak{D}}_{p,p}(E), u_{0,n} \in \mathfrak{D}_p(\mathbf{R}^d) \end{aligned}$$

defined by (13) such that

$$\begin{aligned} &|f_n - f|_{\mathbb{H}_p^\beta(E)} + |\Phi_n - \Phi|_{\mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E)} + |\Phi_n - \Phi|_{\mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} + |h_n - h|_{\mathbb{H}_p^{\beta+\alpha/2}(E)} \\ &+ |g_n - g|_{\mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E)} + |g_n - g|_{\mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} + |u_{0,n} - u_0|_{\mathbb{B}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(\mathbf{R}^d)} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Since  $\Lambda g \in \bar{\mathbb{B}}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E) \cap \bar{\mathbb{H}}_{2,p}^{\beta+\frac{\alpha}{2}}(E)$  it follows by the definition of the approximating sequence that  $\Lambda g_n \in \bar{\mathfrak{D}}_{2,p}(E) \cap \bar{\mathfrak{D}}_{p,p}(E)$  and

$$|\Lambda g_n - \Lambda g|_{\mathbb{H}_{2,p}^{\beta+\frac{\alpha}{2}}(E)} + |\Lambda g_n - \Lambda g|_{\mathbb{B}_{p,pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} \rightarrow 0$$

as  $n \rightarrow \infty$  as well. Since  $I_\varepsilon g \rightarrow Ig$  in  $\mathbb{H}_p^\beta(E)$  as  $\varepsilon \rightarrow 0$  we have for each  $n$  and  $\kappa \in \mathbf{R}$  (see estimate of Lemma 10)

$$(I_\varepsilon g)_n = I_\varepsilon g_n \rightarrow (Ig)_n = Ig_n \text{ as } \varepsilon \rightarrow 0 \text{ in } \mathbb{H}_p^\kappa(E),$$

where  $(I_\varepsilon g)_n$  and  $(Ig)_n$  are approximations defined by (13). In addition, by Lemma 10,

$$(Ig)_n = Ig_n \rightarrow Ig \text{ in } \mathbb{H}_p^\beta(E) \text{ as } n \rightarrow \infty,$$

and for each  $n$  and multiindex  $\gamma \in \mathbf{N}_0^d$

$$\mathbf{E} \left[ \sup_{(s,x) \in E} |D_x^\gamma I_\varepsilon g_n(s, x) - D_x^\gamma Ig_n(s, x)|^p \right] \rightarrow 0$$



as  $\varepsilon \rightarrow 0$ . Therefore all the assumptions of Proposition 21 are satisfied with smooth input functions  $f_n, g_n, h_n, u_{0,n}, \Phi_n$ . Let us denote  $u_n$  the corresponding smooth solution of the class  $\mathfrak{D}_p(E)$ . By definition,

$$\begin{aligned}
& u_n(t) \\
&= u_{n,0} + \int_0^t [A^{(\alpha)} u_n(s) - \lambda u_n(s) + f_n(s)] ds \\
&+ \int_0^t \int_{\mathbf{R}_0^d} [u_n(s-, \cdot + y) - u_n(s-, \cdot) + g_n(s, \cdot, y)] q^{(\alpha)}(ds, dy) 1_{\alpha \in (0,2)} \\
&+ \int_0^t \int_U \Phi_n(s, \cdot, v) \eta(ds, dv) + \int_0^t [1_{\alpha=2} \sigma^i(s) \partial_i u_n(s) + h_n(s)] dW_s, \\
&0 \leq t \leq T.
\end{aligned} \tag{35}$$

According to the estimate of Proposition 21, there is a constant  $C$  independent of  $n, m$  such that

$$\begin{aligned}
|u_n - u_m|_{\mathbb{H}_p^{\beta+\alpha}(E)} &\leq C[|u_{n,0} - u_{m,0}|_{\mathbb{B}_p^{\beta+\alpha-\frac{\alpha}{p}}(\mathbf{R}^d)} + |f_n - f_m - (Ig_n - Ig_m)|_{\mathbb{H}_p^\beta(E)} \\
&+ |\Phi_n - \Phi_m|_{\mathbb{H}_p^{\beta+\frac{\alpha}{2}}(E)} + |\Phi_n - \Phi_m|_{\mathbb{B}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)} \\
&+ |\Lambda g_n - \Lambda g_m|_{\mathbb{H}_p^{\beta+\frac{\alpha}{2}}(E)} + |\Lambda g_n - \Lambda g_m|_{\mathbb{B}_{pp}^{\beta+\alpha-\frac{\alpha}{p}}(E)}].
\end{aligned}$$

Therefore the sequence  $u_n$  is Cauchy in  $\mathbb{H}_p^{\beta+\alpha}(E)$  and there is  $u \in \mathbb{H}_p^{\beta+\alpha}(E)$  such that  $|u_n - u|_{\mathbb{H}_p^{\beta+\alpha}(E)} \rightarrow 0$  as  $n \rightarrow \infty$ . Using Lemmas 12, 17, Corollary 13 and Theorem 1 in [16], we pass easily to the limit in (35) as  $n \rightarrow \infty$  in  $H_p^\beta(\mathbf{R}^d)$ . Obviously,  $u(t)$  is  $H_p^\beta(\mathbf{R}^d)$ -valued cadlag function.

The uniqueness follows by Corollary 22. Theorem 8 is proved.

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